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CHAPTER - 6
APPLICATION OF DERIVATIVES

★ EXERCISE - 6.1

Q1.) Find the rate of change of the area of a circle with respect to its radius r when

(a) r = 3 cm   (b) r = 4 cm

Sol. $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

(a) $\frac{dA}{dr} = 2\pi \times 3 \Rightarrow 6\pi \ cm^2/cm$ Ans

(b) $\frac{dA}{dr} = 2\pi \times 4 \Rightarrow 8\pi \ cm^2/cm$ Ans

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Q2.) The volume of a cube is increasing at the rate of $8 cm^3/s$. How fast is the surface area increasing when the length of an edge is 12 cm?

Sol. $\frac{dV}{dt} = 8 \ cm^3/s$

$\Rightarrow \frac{da^3}{dt} = 8$

$\Rightarrow \frac{da^3}{da} \times \frac{da}{dt} = 8$

$\Rightarrow 3a^2 \times \frac{da}{dt} = 8$

$\Rightarrow \frac{da}{dt} = \frac{8}{3a^2}$

$$\frac{dS}{dt} = \frac{6(da^2)}{dt} \times \frac{da}{da}$$

$$= \frac{6(da^2)}{da} \times \frac{da}{dt}$$

$$= 12a \times \frac{da}{dt}$$

$$= \cancel{12} \times \cancel{12} \times \frac{8}{3 \times \cancel{12} \times \cancel{12}}$$

$$= \frac{8}{3} \text{ cm}^2/\text{s} \quad \underline{\text{Ans}}$$

Q3.) The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.?

Sol. $\frac{dr}{dt} = 3 \text{ cm/s}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{\pi dr^2}{dt} \times \frac{dr}{dr}$$

$$= \pi \frac{dr^2}{dr} \times \frac{dr}{dt}$$

$$= 2\pi r \times 3$$

$$= 6\pi r$$

$$= 6\pi \times 10$$

$$= 60\pi \text{ cm}^2/\text{s} \quad \underline{\text{Ans}}$$

Q4.) An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

$$\frac{dS}{dt} = \frac{d(6a^2)}{dt}$$

$$= \frac{d(6a^2)}{da} \times \frac{da}{dt}$$

$$= 12a \times \frac{da}{dt}$$

Sol. $\frac{da}{dt} = 3 \text{ cm/s}$

$$V = a^3$$

$$\frac{dV}{dt} = \frac{da^3}{dt} \times \frac{da}{da}$$

$$= \frac{da^3}{da} \times \frac{da}{dt}$$

$$= 3a^2 \times 3$$

$$= 3 \times 10 \times 10 \times 3$$

$$= 900 \text{ cm}^3/\text{s}$$ Ans

Q5) A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Sol. $\frac{dr}{dt} = 5 \text{ cm/s}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi \frac{dr^2}{dt} \times \frac{dr}{dr}$$

$$= \pi 2r \times \frac{dr}{dt}$$

$$= \pi \times 2 \times 8 \times 5$$

$= 80\pi \ cm^2/s$ Ans

Q6.) The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Sol. $\frac{dr}{dt} = 0.7 \ cm/s$

$C = 2\pi r$

$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

$= 2\pi \times 0.7$

$= 1.4\pi \ cm/s$ Ans

Q7.) The length x of a rectangle is decreasing at the rate of 5cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Sol. $\frac{dx}{dt} = -5 \ cm/min$, $\frac{dy}{dt} = 4 cm/min$

(a) $P = 2(x+y)$

$\frac{dP}{dt} = 2\left(\frac{d(x+y)}{dt}\right) = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2 \ cm/min$ Ans

(b) $A = xy$

$\frac{dA}{dt} = \frac{d(xy)}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

$= 6 \times -5 + 8 \times 4$

$= -30 + 32$

$= 2 \ cm^2/min$ Ans

Q9.) A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Sol. $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4\pi}{3}\frac{dr^3}{dr}$

$= \frac{4\pi \times \cancel{3}r^2}{\cancel{3}}$

$= 4\pi \times 10 \times 10$

$= 400\pi \ cm^3/cm$ Ans

Q8.) A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Sol. $\frac{dV}{dt} = 900 \ cm^3/s$

$\Rightarrow \frac{dV}{dr} \times \frac{dr}{dt} = 900$

$\Rightarrow \frac{4\pi}{3}\frac{dr^3}{dr} \times \frac{dr}{dt} = 900$

$\Rightarrow \frac{4\pi \times \cancel{3}r^2}{\cancel{3}} \times \frac{dr}{dt} = 900$

$\Rightarrow \frac{dr}{dt} = \frac{\cancel{900}\ \cancel{60}\ 4}{\cancel{4} \times \pi \times \cancel{15} \times \cancel{15}}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \ cm/s$ Ans

Q10.) A ladder 5m long is leaning against a wall. The bottom of the

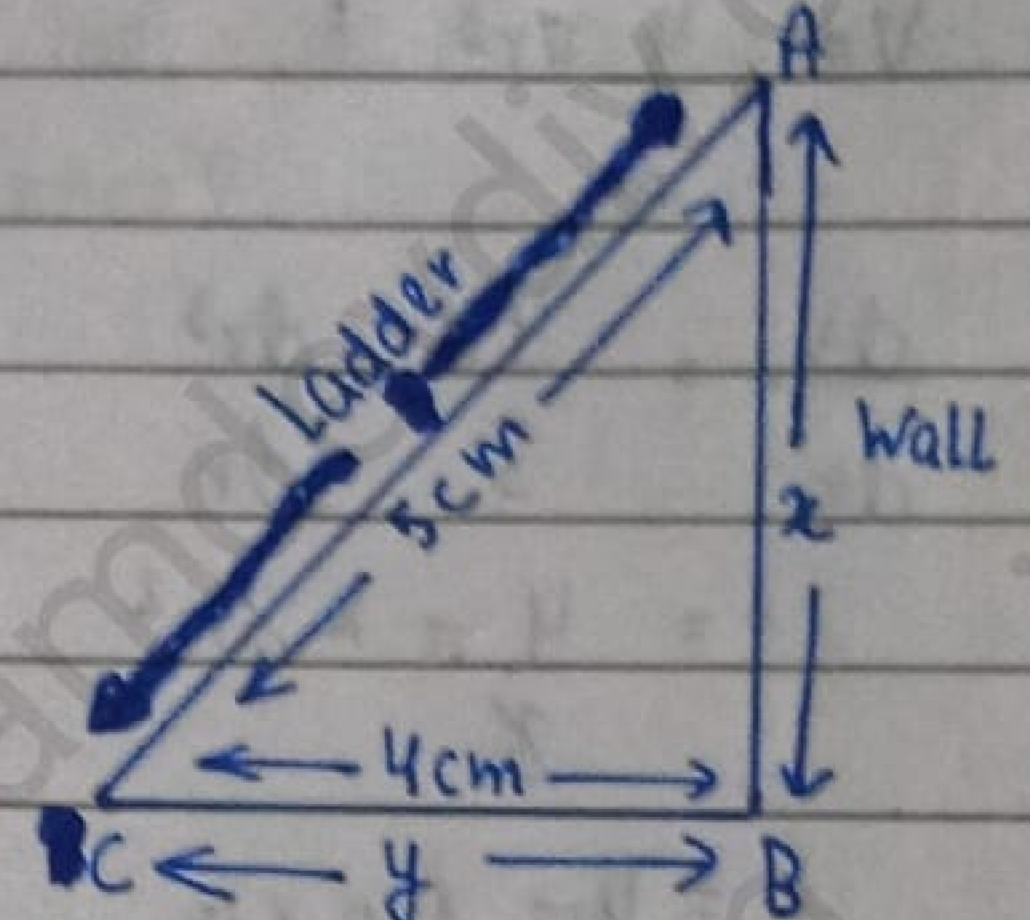
Ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4m away from the wall?

Sol. $(AC)^2 = (AB)^2 + (BC)^2$

$\Rightarrow 25 = x^2 + y^2$ ——— ①

$\Rightarrow 25 = x^2 + 16$

$\Rightarrow x = 3$ cm



Differentiating eq^n w.r.t 't';

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow -\cancel{2}x\frac{dx}{dt} = \cancel{2}y\frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$$

Given, $\frac{dy}{dt} = 2$ cm/s

$$\Rightarrow \frac{dx}{dt} = \frac{-2y}{x}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-2\times 4}{3}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{8}{3} \text{ cm/s}$$

The height on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s Ans

Q11.) A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Sol. $6y = x^3 + 2$

Differentiating w.r.t t,

$$\cancel{8}^{2}\frac{dy}{dt} = \cancel{8}x^2\frac{dx}{dt}$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

Given, $\frac{dy}{dt} = 8\frac{dx}{dt}$

$$\Rightarrow 2\times 8\frac{\cancel{dx}}{\cancel{dt}} = x^2\frac{\cancel{dx}}{\cancel{dt}}$$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

$\underline{x = 4}$

$6y = 64 + 2$

$\Rightarrow y = 11$

$\underline{x = -4}$

$6y = -64 + 2$

$\Rightarrow y = \frac{-31}{3}$

$\therefore$ Points are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$ Ans

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Q12.) The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Sol. $\frac{dr}{dt} = \frac{1}{2}$ cm/s

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\frac{dr^3}{dr}\times\frac{dr}{dt}$$

$$= \frac{4}{\cancel{3}}\pi\times\cancel{3}r^2\times\frac{dr}{dt}$$

$$= 4\pi\times(1)^2\times\frac{1}{2}$$

$= 2\pi \ cm^3/s$ Ans

Q13.) A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Sol. Diameter $= \frac{3}{2}(2x+1)$

Radius $= \frac{3}{4}(2x+1)$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3 = \frac{\cancel{4}}{\cancel{3}}\pi \frac{\cancel{27}^{9}}{\cancel{64}_{16}}(2x+1)^3$

$\frac{dV}{dx} = \frac{9}{\cancel{16}8}\pi \times 3(2x+1)^2 \times \cancel{2}$

$= \frac{27}{8}\pi(2x+1)^2$ Ans

Q14.) Sand is pouring from a pipe at the rate of $12 cm^3/s$. The falling sand forms a cone on the ground in such a way that height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Sol. $\frac{dV}{dt} = 12 \ cm^3/s$

$h = \frac{r}{6} \Rightarrow r = 6h$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{\cancel{3}}\pi \times \cancel{6}^{2}h \times 6h \times h = 12\pi h^3$

Derivating with respect to t,

$\frac{dV}{dt} = 12\pi \frac{dh^3}{dh}\frac{dh}{dt}$

$$\Rightarrow \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \cancel{12} = \cancel{36}^{3}\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi \times 4 \times 4}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{48\pi} \quad \text{Ans}$$

Q15.) The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.

Sol. $C'(x) = 0.021x^2 - 0.006x + 15$

$C'(17) = 0.021 \times 17 \times 17 - 0.006 \times 17 + 15$

$= 6.069 - 0.102 + 15$

$= 21.069 - 0.102$

$= ₹\,20.967$ Ans

Q16.) The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Sol. $R'(x) = 26x + 26$

$R'(7) = 26 \times 7 + 26$

$= 182 + 26$

$= ₹\,208$ Ans

- Choose the correct answer for questions 17 and 18.

Q17.) The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is

(A) 10π (B) 12π (C) 8π (D) 11π

Sol. $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r = 2 \times \pi \times 6 = 12\pi$$

$\therefore$ (B) 12π Ans

Q18.) The total revenue in Rupees received from the sale of x units of a product is given by

$R(x) = 3x^2 + 36x + 5.$

The marginal revenue, when $x = 15$ is

(A) 116 (B) 96 (C) 90 (D) 126

Sol. $R'(x) = 6x + 36$

$R'(15) = 90 + 36 = 126$

$\therefore$ (D) 126 Ans

EXTRA QUESTION

Q. Find the rate of change of TSA of a cylinder of the radius r and height h and radius varies.

Sol. $TSA = A = 2\pi r(r+h)$

$$\frac{dA}{dr} = (r+h)\frac{d}{dr}(2\pi r) + 2\pi r\frac{d}{dr}(r+h)$$

$= (r+h)2\pi + 2\pi r$

$= 2\pi r + 2\pi h + 2\pi r$

$= 2\pi h + 4\pi r$

$= 2\pi(h+2r)$ Ans

EXTRA QUESTION

Q The volume of a cube is increasing at a constant rate. Prove that the increase in ~~SA~~ surface area varies inversely as the length of the edge of cube.

Sol. $\frac{dV}{dt} = k$

$\Rightarrow \frac{da^3}{da} \times \frac{da}{dt} = k$

$\Rightarrow 3a^2 \frac{da}{dt} = k$

$\Rightarrow \frac{da}{dt} = \frac{k}{3a^2}$

$S = 6a^2$

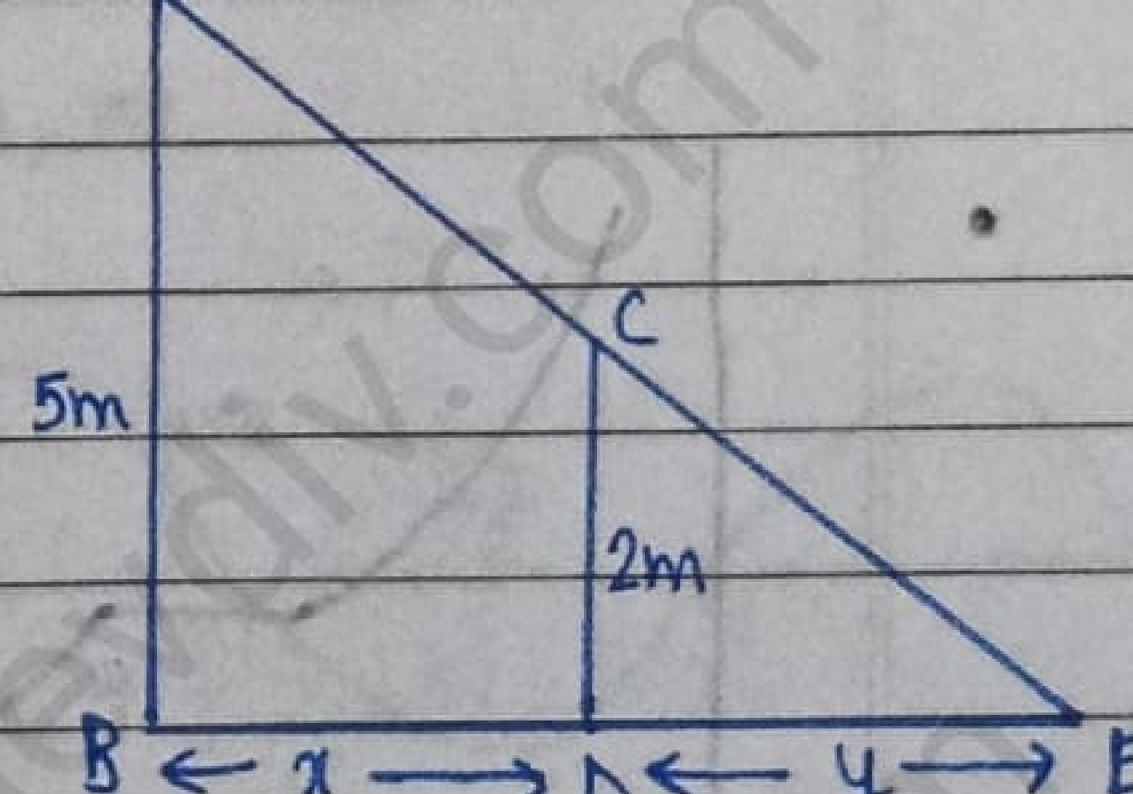
$\frac{dS}{dt} = 6\frac{da^2}{da} \times \frac{da}{dt} = 12a \frac{da}{dt} = \cancel{12}^{4}\cancel{a} \times \frac{k}{\cancel{3}a^{\cancel{2}}} = \frac{4k}{a}$

$\frac{dS}{dt} \propto \frac{1}{a}$

Hence, proved

EXTRA QUESTION

Q A man 2m high walks at a uniform speed of 6m/min away from a lamp post 5m high. Find the rate at which length of his shadow increases.



Sol. $\frac{dx}{dt} = 6\,m/min$

$\frac{AB}{CD} = \frac{BE}{DE} \Rightarrow \frac{5}{2} = \frac{x+y}{y}$

$\Rightarrow 5y = 2x + 2y$

$\Rightarrow y = \frac{2}{3}x$

$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3} \times 6 = 4\,m/min$ Ans

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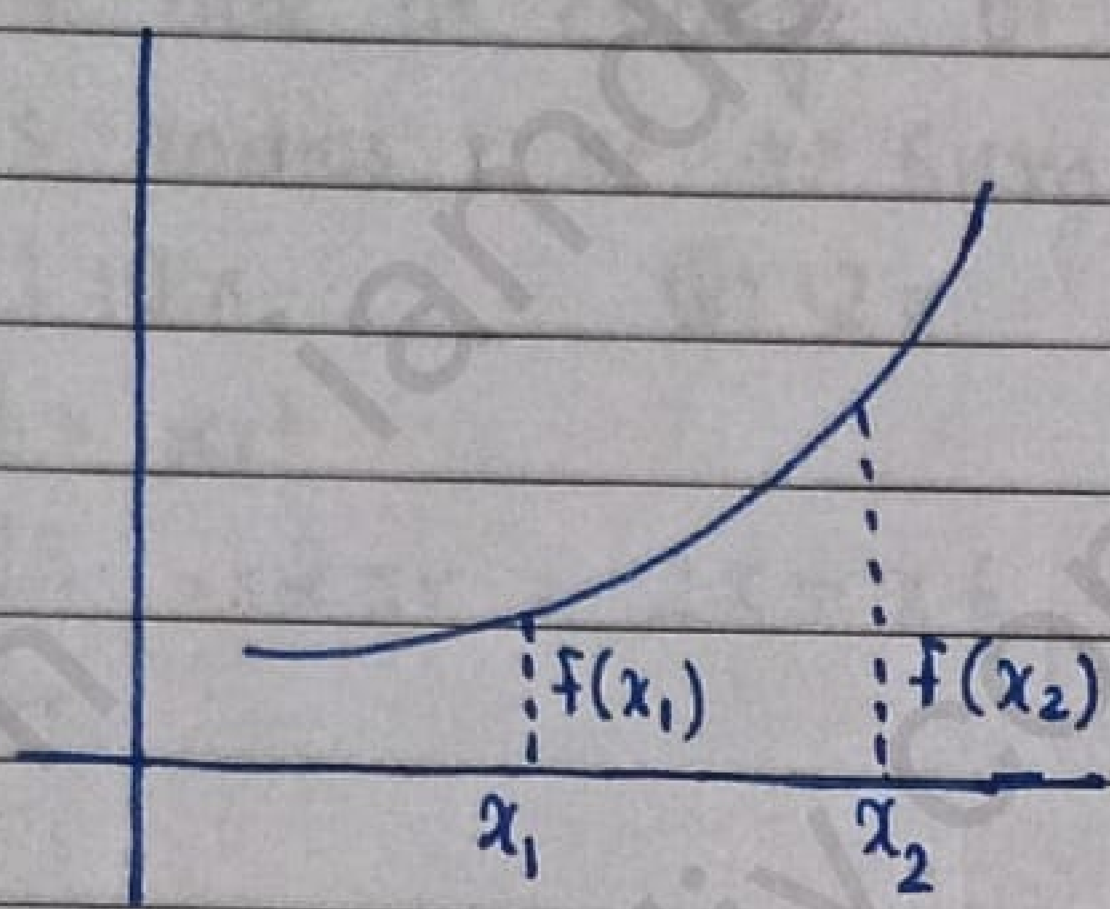
☆ INCREASING AND DECREASING FUNCTIONS

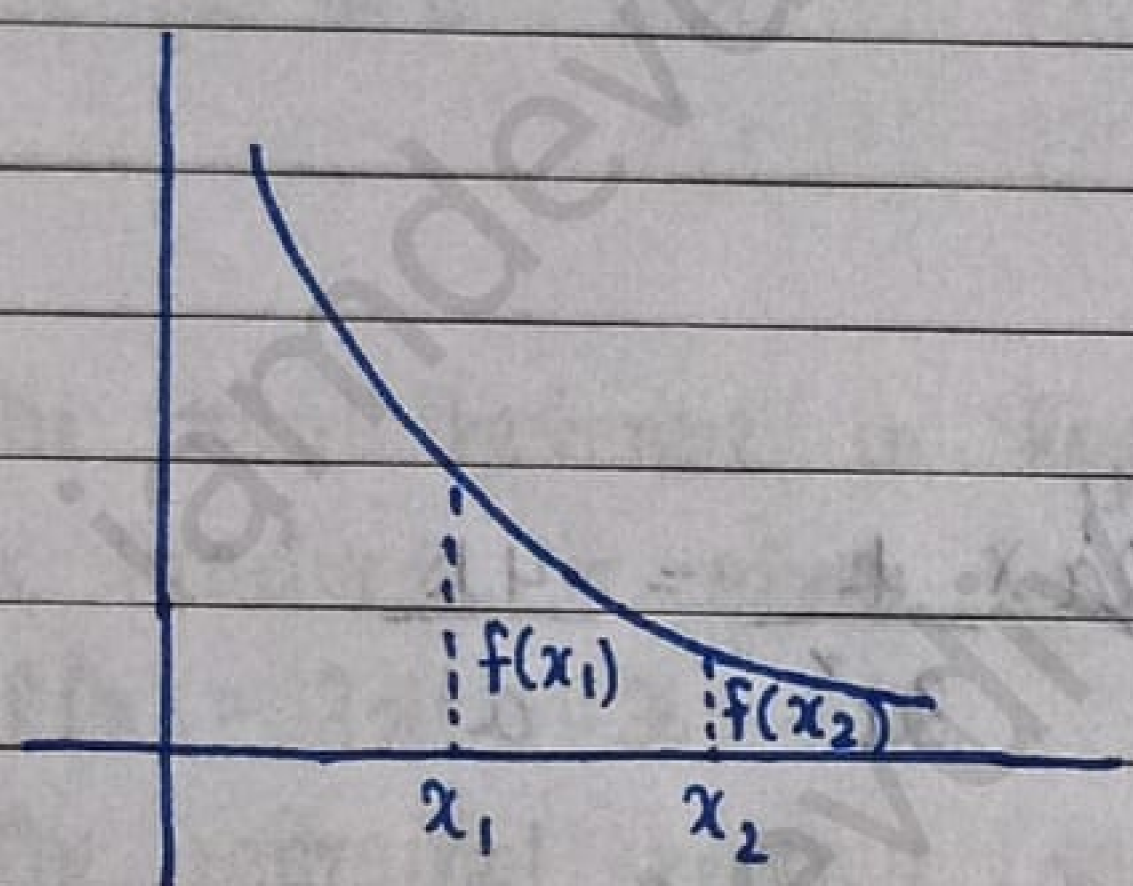
- 

  $x_1 < x_2$ $\quad [x_1, x_2 \in R]$

  $f(x_1) < f(x_2)$

  f is strictly increasing

  $f'(x) > 0$

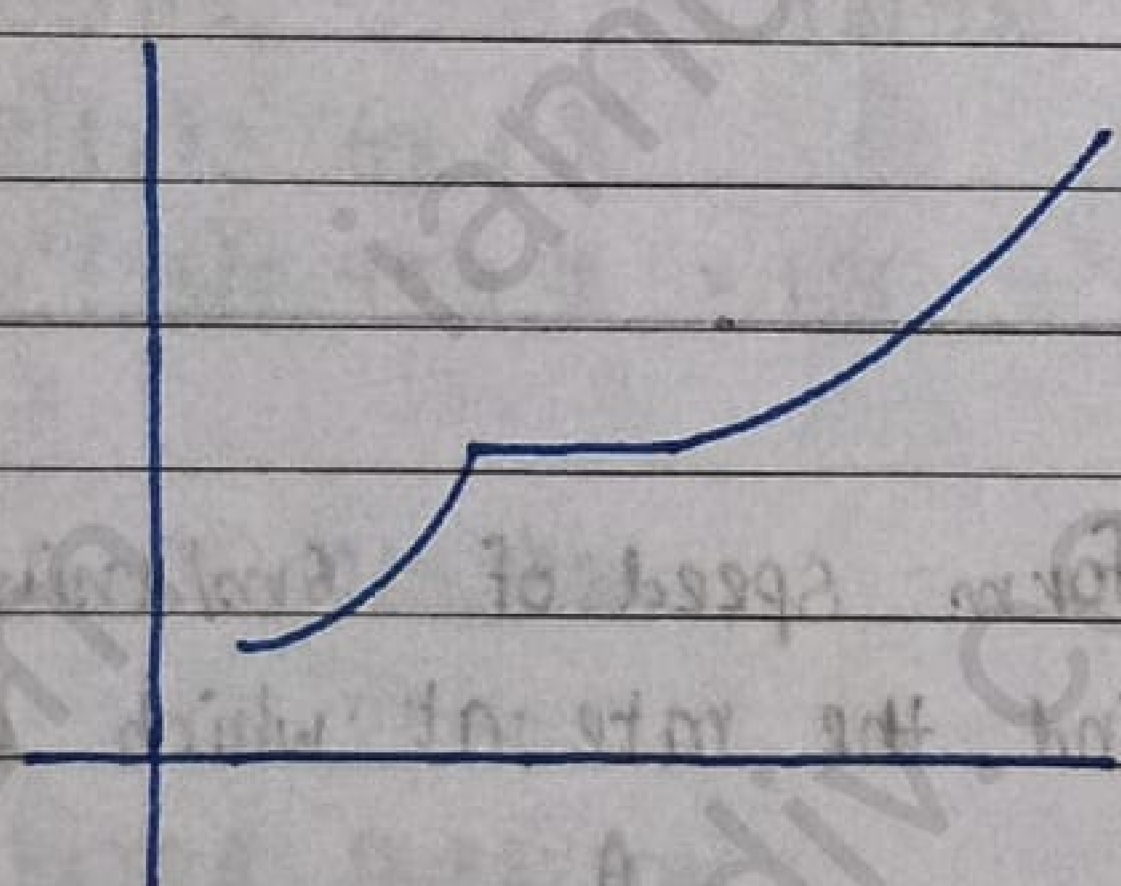
- 

  $x_1 < x_2$ $\quad [x_1, x_2 \in R]$

  $f(x_1) > f(x_2)$

  f is strictly decreasing

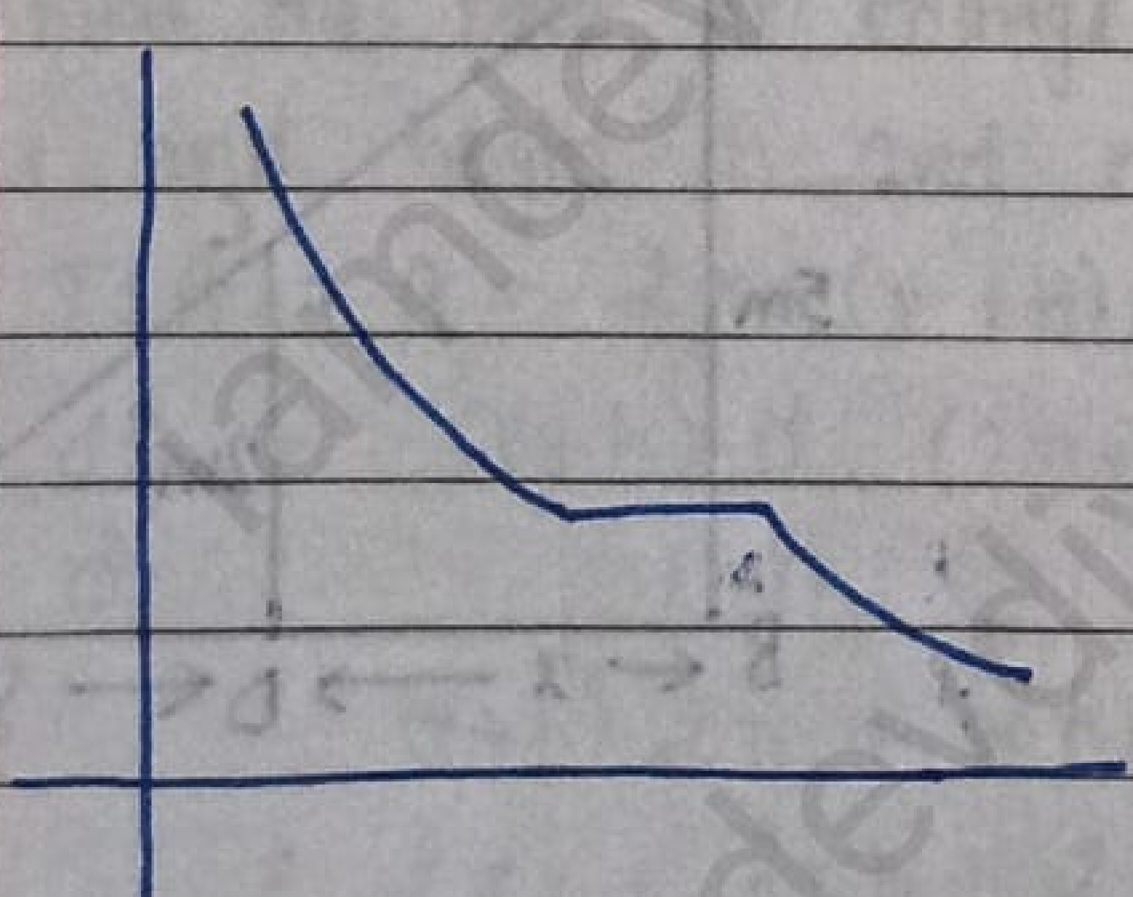
  $f'(x) < 0$

- 

  $x_1 \leq x_2$

  $f(x_1) \leq f(x_2)$

  f is increasing

- 

  $x_1 \leq x_2$

  $f(x_1) \geq f(x_2)$

  f is decreasing

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☆ EXERCISE – 6.2

Q1.) Show that the function given by $f(x) = 3x + 17$ is increasing on R.

Sol. $f'(x) = 3$

$f'(x) > 0$

$\therefore f$ is increasing on R

Q2.) Show that the function given by $f(x) = e^{2x}$ is increasing on R.

Sol. Let $x_1, x_2 \in R$

such that $x_1 < x_2$

$\Rightarrow 2x_1 < 2x_2$

$\Rightarrow e^{2x_1} < e^{2x_2}$

$\Rightarrow f(x_1) < f(x_2)$

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\therefore f$ is increasing on R

Q3.) Show that the function given by $f(x) = \sin x$ is

(a) increasing in $\left[0, \frac{\pi}{2}\right]$ (b) decreasing in $\left[\frac{\pi}{2}, \pi\right]$

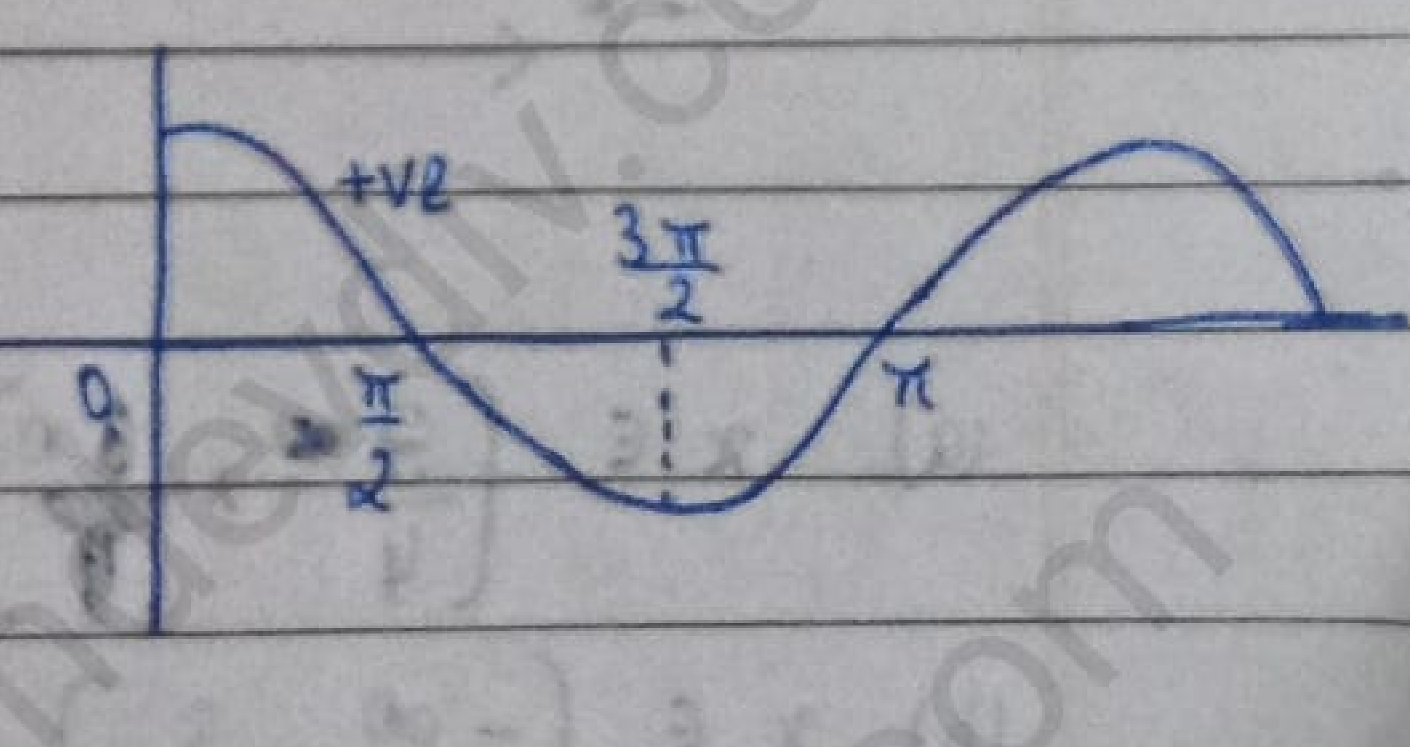
(c) neither increasing nor decreasing in $(0, \pi)$

Sol. (a) $f(x) = \sin x$

$f'(x) = \cos x$

$f'(x) > 0$ for $x \in \left[0, \frac{\pi}{2}\right)$

$\therefore f(x)$ is increasing in $\left[0, \frac{\pi}{2}\right]$



(b) $f(x) = \sin x$

$f'(x) = \cos x$

$f'(x) < 0$ for $x \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore$ $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) $f'(x) < 0$ for $x \in \left(\frac{\pi}{2}, \pi\right)$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) > 0$ and $f'(x) < 0$ for $x \in (0, \pi)$

$\therefore$ $f(x)$ is neither decreasing nor increasing in $(0, \pi)$

Qu.) Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is
(a) increasing (b) decreasing

Sol. $f(x) = 2x^2 - 3x$

$f'(x) = 4x - 3$

$f'(x) = 0$

$\Rightarrow 4x - 3 = 0$

$\Rightarrow x = \frac{3}{4}$

$-\infty \longleftarrow \frac{3}{4} \longrightarrow \infty$

(a) $x \in \left(\frac{3}{4}, \infty\right) \Rightarrow f$ is increasing

(b) $x \in \left(-\infty, \frac{3}{4}\right) \Rightarrow f$ is decreasing

Q5.) Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) increasing (b) decreasing

Sol. $f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36$

$f'(x) = 0$

$\Rightarrow 6x^2 - 6x - 36 = 0$

$\Rightarrow 6(x^2 - x - 6) = 0$

$\Rightarrow 6(x-3)(x+2) = 0$

$x = 3, -2$

$-\infty \leftarrow \quad -2 \quad 3 \quad \rightarrow \infty$

$x \in (-\infty, -2)$ $f'(x) > 0$

$x \in (-2, 3)$ $f'(x) < 0$

$x \in (3, \infty)$ $f'(x) > 0$

(a) $x \in (-\infty, -2) \cup (3, \infty)$

f is increasing

(b) $x \in (-2, 3)$

f is decreasing

Q6.) Find the intervals in which the following functions are strictly increasing or decreasing

(a) $x^2 + 2x - 5$

(a) $f(x) = x^2 + 2x - 5$

$f'(x) = 2x + 2$

$f'(x) = 0$

$\Rightarrow 2(x+1) = 0$

$x = -1$

$$-\infty \leftarrow \underset{-1}{+} \rightarrow \infty$$

$x \in (-\infty, -1) \Rightarrow f'(x) < 0 \Rightarrow f$ is decreasing
$x \in (-1, \infty) \Rightarrow f'(x) > 0 \Rightarrow f$ is increasing

(b) $10 - 6x - 2x^2$

(b) $f(x) = 10 - 6x - 2x^2$
$f'(x) = -6 - 4x$

$f'(x) = 0$
$\Rightarrow -6 - 4x = 0$
$\Rightarrow -2(3 + 2x) = 0$
$x = -\frac{3}{2}$

$$-\infty \leftarrow \underset{-3/2}{+} \rightarrow \infty$$

$x \in \left(-\infty, -\frac{3}{2}\right) \Rightarrow f'(x) > 0 \Rightarrow f$ is increasing

$x \in \left(\frac{-3}{2}, \infty\right) \Rightarrow f'(x) < 0 \Rightarrow f$ is decreasing

(c) $-2x^3 - 9x^2 - 12x + 1$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$
$f'(x) = -6x^2 - 18x - 12$

$f'(x) = 0$
$\Rightarrow -6(x^2 + 3x + 2) = 0$
$\Rightarrow -6(x+1)(x+2) = 0$
$x = -1, x = -2$

$$-\infty \leftarrow \underset{-2}{+} \quad \underset{-1}{+} \rightarrow \infty$$

$x \in (-\infty, -2)$ $f'(x) > 0$

$x \in (-2, -1) \qquad f'(x) < 0$

$x \in (-1, \infty) \qquad f'(x) > 0$

$x \in (-\infty, -2) \cup (-1, \infty)$

f is increasing

$x \in (-2, -1)$

f is decreasing

(d) $6 - 9x - x^2$

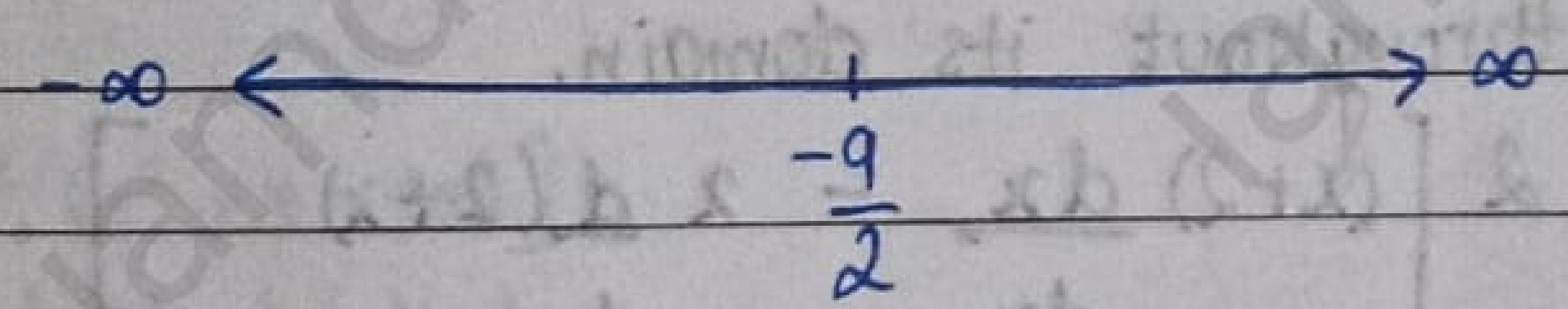
(d) $f(x) = 6 - 9x - x^2$

$f'(x) = -9 - 2x$

$f'(x) = 0$

$\Rightarrow -9 = 2x$

$\Rightarrow x = \frac{-9}{2}$



$x \in \left[-\infty, \frac{-9}{2}\right] \Rightarrow f'(x) > 0 \Rightarrow$ f is increasing

$x \in \left[\frac{-9}{2}, \infty\right] \Rightarrow f'(x) < 0 \Rightarrow$ f is decreasing

(e) $(x+1)^3 (x-3)^3$

(e) $f(x) = (x+1)^3 (x-3)^3$

$f'(x) = (x+1)^3 \, 3(x-3)^2 + (x-3)^3 \, 3(x+1)^2$

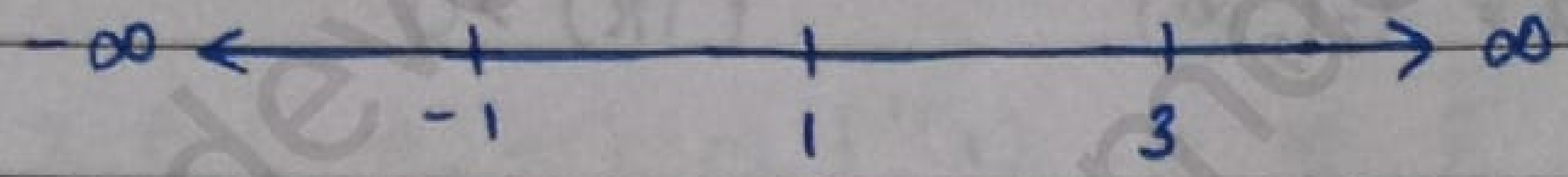
$= 3(x+1)^2 (x-3)^2 (2x-2)$

$= 6(x+1)^2 (x-3)^2 (x-1)$

$f'(x) = 0$

$\Rightarrow 6(x+1)^2 (x-3)^2 (x-1) = 0$

$x = -1, 3, 1$



$x \in (-\infty, -1)$ $\quad f'(x) < 0$

$x \in (-1, 1)$ $\quad f'(x) < 0$

$x \in (1, 3)$ $\quad f'(x) > 0$

$x \in (3, \infty)$ $\quad f'(x) > 0$

$x \in (-\infty, -1) \cup (-1, 1)$

f is decreasing

$x \in (1, 3) \cup (3, \infty)$

f is increasing

Q7.) Show that $y = \log(1+x) - \dfrac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

Sol. $\dfrac{dy}{dx} = \dfrac{1}{1+x} - 2\left[\dfrac{(x+2)\frac{dx}{dx} - x\frac{d(2+x)}{dx}}{(x+2)^2}\right]$

$= \dfrac{1}{1+x} - \left[\dfrac{2x - 4 - 2x}{(x+2)^2}\right]$

$= \dfrac{1}{1+x} - \dfrac{4}{(x+2)^2}$

$= \dfrac{(x+2)^2 - 4(1+x)}{(x+1)(x+2)^2}$

$= \dfrac{x^2 + 4 + 4x - 4 - 4x}{(x+1)(x+2)^2}$

$= \dfrac{x^2}{(x+1)(x+2)^2}$

$\dfrac{dy}{dx} > 0$ ~~$\therefore$ f(x) is increasing~~

$\Rightarrow \dfrac{x^2}{(x+1)(x+2)^2} > 0$

$\Rightarrow \left[\dfrac{x}{(x+2)}\right]^2 \times \dfrac{1}{x+1} > 0$

$\because \left(\dfrac{x}{x+2}\right)^2 > 0$

$\dfrac{1}{x+1} > 0$

$\Rightarrow 1 > 0$

$x+1 > 0$

$x > -1$

$\therefore$ y is an increasing function of x throughout its domain.

Q8) Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

Sol. $y = [x^2(x^2+4-4x)]$

$\Rightarrow y = x^4 + 4x^2 - 4x^3$

$\dfrac{dy}{dx} = 4x^3 + 8x - 12x^2$

$\Rightarrow \dfrac{dy}{dx} = 4x(x^2 - 3x + 2)$

$\Rightarrow \dfrac{dy}{dx} = 4x(x^2 - 2x - x + 2)$

$\Rightarrow \dfrac{dy}{dx} = 4x(x(x-2) - 1(x-2))$

$\Rightarrow \dfrac{dy}{dx} = 4x(x-2)(x-1)$

$\dfrac{dy}{dx} = 0$

$x = 0, 2, 1$

$-\infty \leftarrow \quad 0 \quad 1 \quad 2 \quad \rightarrow \infty$

$x \in (-\infty, 0)$ $\quad f'(x) < 0$

$x \in (0, 1)$ $\quad f'(x) > 0$

$x \in (1, 2)$ $\quad f'(x) < 0$

$x \in (2, \infty)$ $\quad f'(x) > 0$

for $x \in (0,1) \cup (2, \infty)$,
f is increasing

Q9.) Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Sol. $\frac{dy}{d\theta} = \frac{(2+\cos\theta)\frac{d}{d\theta}(4\sin\theta) - 4\sin\theta\frac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - \frac{d\theta}{d\theta}$

$= \frac{(2+\cos\theta)4\cos\theta + 4\sin\theta\sin\theta}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$

$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$

$= \frac{8\cos\theta + \cancel{4} - \cancel{4} - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$

$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$

$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$

$\theta \in \left[0, \frac{\pi}{2}\right]$

$0 \le \cos\theta \le 1$

$4 - \cos\theta > 1$

$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0 \quad \theta \in \left[0, \frac{\pi}{2}\right]$

Hence, proved

Q10.) Prove that the ~~logra~~ Logarithmic function is increasing on $(0, \infty)$.

Sol. $f(x) = \log x \quad x \in (0, \infty)$

$f'(x) = \frac{1}{x} > 0 \quad x \in (0, \infty)$

Q11.) Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(-1, 1)$.

Sol. $f(x) = x^2 - x + 1$

$f'(x) = 2x - 1$

$f'(x) = 0$

$x = \frac{1}{2}$

$-1 \qquad \frac{1}{2} \qquad 1$

$x \in (-1, \frac{1}{2}) \quad f'(x) < 0$

$x \in (\frac{1}{2}, 1) \quad f'(x) > 0$

$x \in (-1, 1)$

f is neither strictly increasing nor decreasing

Hence, proved

Q12.) Which of the following functions are decreasing on $\left[0, \frac{\pi}{2}\right]$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

Sol. (A) $f(x) = \cos x$

$f'(x) = -\sin x$

$f'(x) < 0 \quad x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \cos x$ is decreasing function on $\left(0, \frac{\pi}{2}\right)$

(B) $f(x) = \cos 2x$

$f'(x) = -2\sin(2x)$

$f'(x) < 0 \quad x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \cos 2x$ is decreasing function on $\left(0, \frac{\pi}{2}\right)$

(C) $f(x) = \cos 3x$

$f'(x) = -3\sin 3x$

$f'(x)$ is neither positive nor negative on $\left(0, \frac{\pi}{2}\right)$

$\therefore \cos 3x$ is not decreasing function on $\left(0, \frac{\pi}{2}\right)$

(D) $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f'(x) > 0 \quad x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \tan x$ is not decreasing function on $\left(0, \frac{\pi}{2}\right)$

Q13.) On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ decreasing

(A) $(0, 1)$ (B) $\left[\frac{\pi}{2}, \pi\right]$ (C) $\left[0, \frac{\pi}{2}\right]$ (D) None of these

Sol. $f'(x) = 100x^{99} + \cos x$

(A) $x \in (0,1)$

$f'(x) > 0$

f is increasing

(B) $x \in \left[\frac{\pi}{2}, \pi\right]$

$f'(x) = 100\left(\frac{5\pi}{6}\right)^{99} + \cos\left(\frac{5\pi}{6}\right)$

$= 100\left(\frac{5\pi}{6}\right)^{99} + \cos\left(\pi - \frac{\pi}{6}\right)$

$= 100\left(\frac{5\pi}{6}\right)^{99} + \cos\frac{\pi}{6}$

$f'(x) > 0$

f is increasing

(C) $x \in \left[0, \frac{\pi}{2}\right]$

$f'(x) = 100\left(\frac{\pi}{4}\right)^{99} + \cos\frac{\pi}{4}$

$f'(x) > 0$

f is increasing

$\therefore$ (D) None of these Ans

Q14.) For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on $[1,2]$?

Sol. $f(x) = x^2 + ax + 1$

$f'(x) = 2x + a$

$\cancel{x \in [1,2]}$

$\cancel{x = 1}$

$x \in [1,2]$

$x = 1$	$x = 2$
$1 = \frac{-a}{2}$	$2 = \frac{-a}{2}$
$\Rightarrow a = -2$	$\Rightarrow a = -4$

$f'(x) > 0$ on $[1,2]$ [Given]

$2x + a > 0$

$x = 1$	$x = 2$
$2(1) + a > 0$	$2(2) + a > 0$
$\Rightarrow a > -2$	$\Rightarrow a > -4$

For $a > -2$, $f(x)$ is increasing on $[1,2]$

Q15) Let I be any interval disjoint from $[-1,1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.

Sol. $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(x) = 0$

$\Rightarrow 1 - \frac{1}{x^2} = 0$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

$-\infty \longleftarrow -1 \quad 1 \longrightarrow \infty$

$x \in (-\infty, -1)$ $\quad f'(x) > 0$

$x \in (-1, 1)$ $\quad f'(x) < 0$

$x \in (1, \infty)$ $\quad f'(x) > 0$

On $x \in (-\infty, -1) \cup (1, \infty)$

$f'(x) \geq 0$

f is increasing

Hence, proved

Q16.) Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Sol. $f(x) = \log \sin x$

$f'(x) = \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x}$

$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos x > 0$ and $\sin x > 0$

$\Rightarrow f'(x) = \frac{\cos x}{\sin x} > 0$

$\Rightarrow f(x)$ is increasing on $x \in \left(0, \frac{\pi}{2}\right)$

$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \cos x < 0$ and $\sin x > 0$

$\Rightarrow f'(x) = \frac{\cos x}{\sin x} < 0$

$\Rightarrow f(x)$ is decreasing on $x \in (\pi/2, \pi)$ Hence, proved

Q17.) Prove that the function f given by $f(x) = \log(\cos x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.

Sol. $f(x) = \log \cos x$

$f'(x) = \frac{1}{\cos x} \times -\sin x = \frac{-\sin x}{\cos x} = -\tan x$

$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow -\tan x < 0$

$\Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$

$x \in \left(\frac{3\pi}{2}, 2\pi\right) \Rightarrow -\tan x > 0$

$\Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$

Hence, proved

Q18.) Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Sol. $f'(x) = 3x^2 - 6x + 3$

$f'(x) = 0$

$\Rightarrow 3(x^2 - 2x + 1) = 0$

$\Rightarrow 3(x - 1)^2 = 0$

$3 > 0$

$(x-1)^2 > 0$

$3(x-1)^2 > 0$

$f'(x) > 0$

$\therefore f(x)$ is increasing in R
Hence, proved

Q19.) The interval in which $y = x^2 e^{-x}$ is increasing is
(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

Sol. $\frac{dy}{dx} = e^{-x} \times 2x + x^2 \times e^{-x} \times (-1)$

$= x e^{-x}(2 - x)$

$(-\infty, \infty) \Rightarrow$ Let $x = -3 \Rightarrow y' < 0$ (decreasing)
$(-2, 0) \Rightarrow$ Let $x = -1 \Rightarrow y' < 0$ (decreasing)
$(2, \infty) \Rightarrow$ Let $x = 3 \Rightarrow y' < 0$ (decreasing)
$(0, 2) \Rightarrow$ Let $x = 1 \Rightarrow y' > 0$ (increasing)

$\therefore$ (D) $(0, 2)$ Ans

EXAMPLE 12

Q Find intervals in which of the ~~following~~ function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is
(a) increasing (b) decreasing

Sol. $f(x) = \sin 3x$
$f'(x) = 3\cos 3x$

$f'(x) = 0$
$\Rightarrow 3\cos 3x = 0$
$\Rightarrow \cos 3x = 0$
$\Rightarrow 3x = \frac{\pi}{2}, \frac{3\pi}{2}$
$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$

(a) $x \in \left[0, \frac{\pi}{6}\right]$ $\quad f'(x) > 0$

f is increasing

(b) $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right]$ $\quad f'(x) < 0$

f is decreasing

EXAMPLE 13

Q Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x \quad , \quad 0 \le x \le 2\pi$$

is increasing or decreasing.

SOL. $f(x) = \sin x + \cos x$

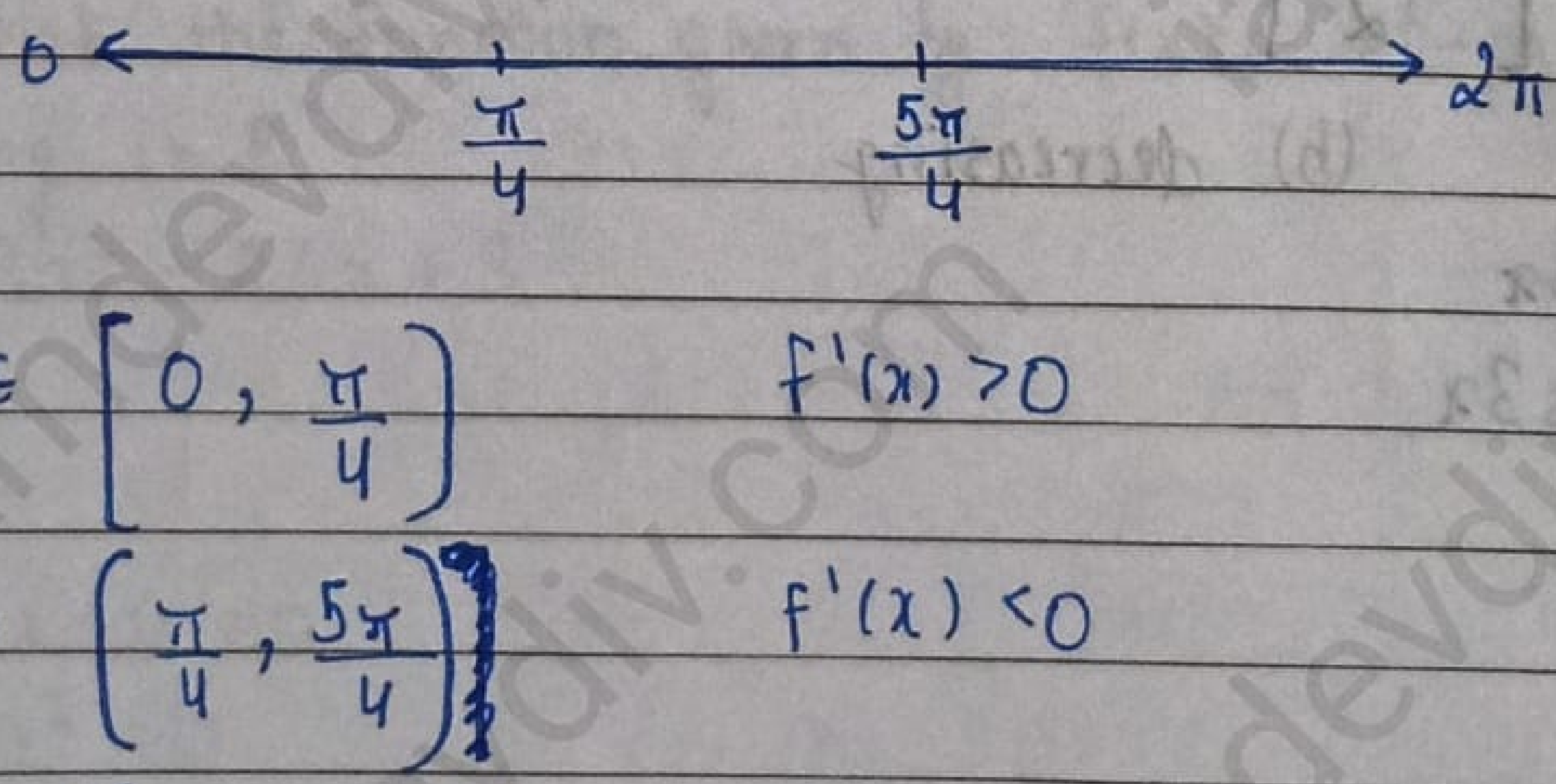
$f'(x) = \cos x - \sin x$

$f'(x) = 0$

$\Rightarrow$ $\cos x = \sin x$

$0 \le x \le 2\pi$

When $x = \frac{\pi}{4}, \frac{5\pi}{4} \Rightarrow \cos x = \sin x$



$x \in \left[0, \frac{\pi}{4}\right)$ $\quad f'(x) > 0$

$x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ $\quad f'(x) < 0$

$x \in \left(\frac{5\pi}{4}, 2\pi\right]$ $\quad f'(x) > 0$

$x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right] \Rightarrow$ f is increasing

$x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \Rightarrow f$ is decreasing

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EXEMPLAR

Q $f(x) = \tan^{-1}(\sin x + \cos x) \qquad x \in \left(0, \frac{\pi}{4}\right)$

Prove $f(x)$ is increasing.

Sol. $f'(x) = \dfrac{1}{1+(\sin x + \cos x)} \times (\cos x - \sin x)$

$= \dfrac{\cos x \cancel{1} - \sin x}{1 + \sin^2 x + \cos^2 x + 2\sin x \cos x}$

$= \dfrac{\cos x - \sin x}{\sin 2x + 2}$

$x \in \left(0, \frac{\pi}{4}\right)$

$\cos x > \sin x$

$\cos x - \sin x > 0$

$\sin 2x + 2 > 0$

$\dfrac{\cos x - \sin x}{\sin 2x + 2} > 0$

$f'(x) > 0$

$\therefore f(x)$ is increasing on $x \in \left(0, \frac{\pi}{4}\right)$

Hence, proved

EXTRA QUESTION

Q $f(x) = \log(1+x) - \dfrac{x}{1+x}$, find the intervals in which function is increasing and decreasing.

Sol. $f(x) = \log(1+x) - \dfrac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{1+\cancel{x}-\cancel{x}}{(1+x)^2}$$

$$= \frac{(\cancel{1}+x)-\cancel{1}}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2}$$

$f'(x) > 0$

$x > 0$

$(1+x)^2 > 0 \Rightarrow$ Domain: $(-1, \infty)$

$x \in (0, \infty)$

$f'(x) < 0$

$x < 0$

$(1+x)^2 > 0 \Rightarrow$ Domain: $(-1, \infty)$

$x \in (-1, 0)$

$f(x)$ is increasing on $(0, \infty)$
and decreasing on $(-1, 0)$

18/8/23

☆ MAXIMA AND MINIMA

• ABSOLUTE OR GLOBAL MAXIMA AND MINIMA

Let f be a function defined on an interval I

(i) f is said to have a maximum value or absolute ~~value~~ maximum or global maximum in I
if there exists a point $c \in I$
such that $f(c) \geq f(x) \quad \forall\, x \in I$
$f(c)$ is called maximum value and c is called maximum point.

(ii) f is said to have a minimum value or absolute minimum or global minimum in I
if there exists a point $c \in I$
such that $f(c) \leq f(x) \quad \forall\, x \in I$
$f(c)$ is called minimum value and c is called minimum point

(iii) f has an extreme value in the interval I
if there exists a point $c \in I$
such that $f(c)$ is either a maximum value or minimum value

Q • Find maximum and minimum values of following functions

(a) $f(x) = |x|, \quad x \in [2, 5]$

(a) $2 \leq x \leq 5$

$\Rightarrow |2| \leq |x| \leq |5|$

$\Rightarrow 2 \leq |x| \leq 5$

$\Rightarrow 2 \leq f(x) \leq 5$

Minimum value $\Rightarrow 5$, Maximum value $\Rightarrow 2$

(b) $f(x) = |x|$, $x \in [-2, 1]$

(b) $-2 \le x \le 1$

$\Rightarrow 0 \le |x| \le 2$

Maximum value $= 2$, Minimum value $= 0$

(c) $f(x) = |2x + 3|$, $x \in [-7, -2]$

(c) $-7 \le x \le -2$

$\Rightarrow -14 \le 2x \le -4$

$\Rightarrow -11 \le 2x + 3 \le -1$

$\Rightarrow 1 \le |2x + 3| \le 11$

Maximum value $= 11$, Minimum value $= 1$

(d) $f(x) = |\sin 4x + 3|$, $x \in R$

(d) $-1 \le \sin x \le 1$

$\Rightarrow -1 \le \sin 4x \le 1$

$\Rightarrow 2 \le \sin 4x + 3 \le 4$

$\Rightarrow 2 \le |\sin 4x + 3| \le 4$

Minimum value $= 2$, Maximum value $= 4$

(e) $f(x) = 2x^3 - 15x^2 + 36x + 1$, $x \in [1, 5]$

(e) $f'(x) = 6x^2 - 30x + 36$

$f'(x) = 0$

$\Rightarrow 6(x^2 - 5x + 6) = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x-3)(x-2) = 0$

$x = 2, 3$

$f(1) = 2 - 15 + 36 + 1 = 24$

$f(2) = 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 1$

$= 16 - 60 + 72 + 1$

$= 29$

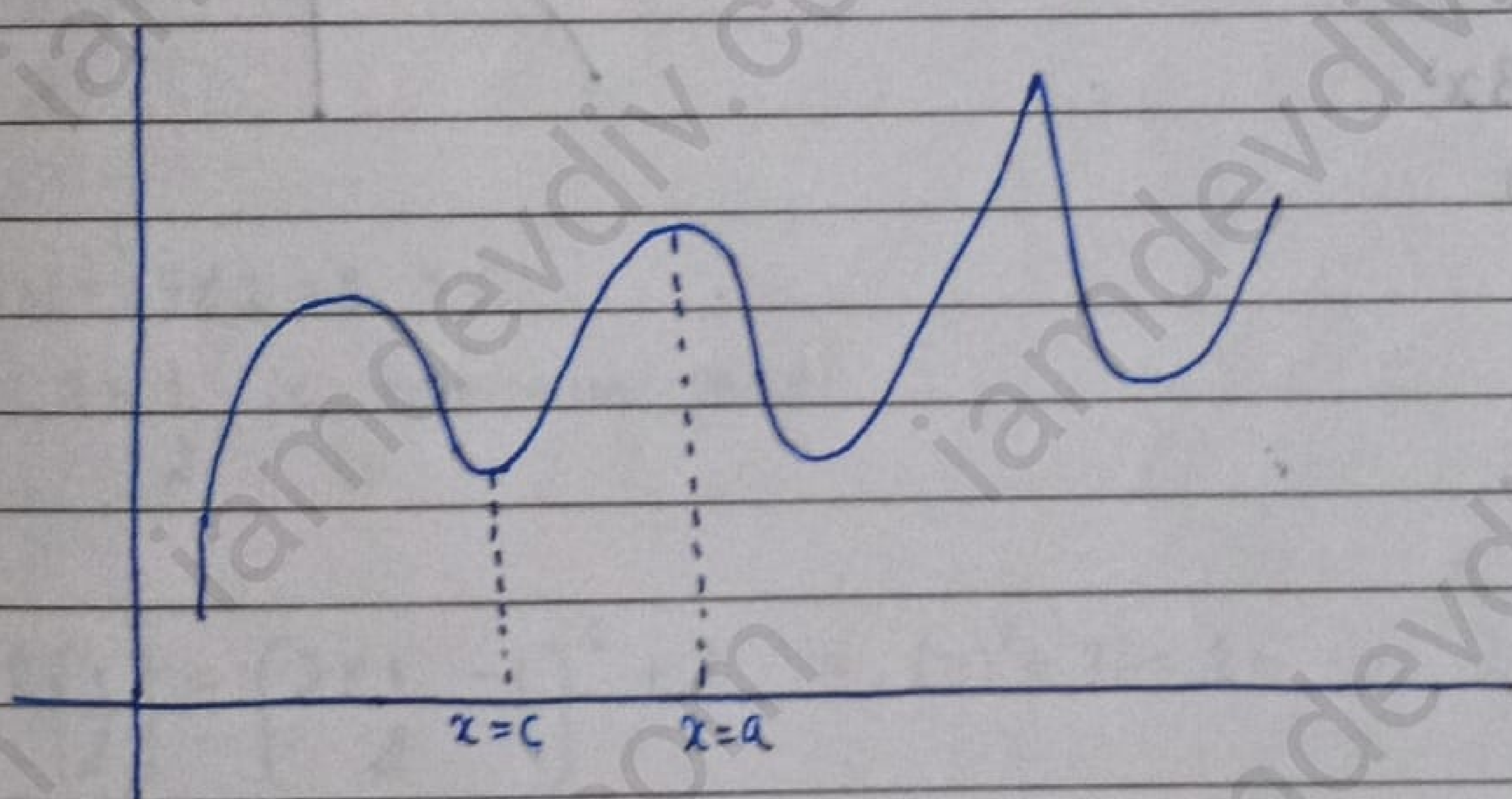
$f(3) = 2 \times 3^3 - 15 \times 2^2 + 36 \times 3 + 1$
$= 54 - 135 + 108 + 1$
$= 28$

$f(5) = 2 \times 5^3 - 15 \times 5^2 + 36 \times 5 + 1$
$= 250 - 315 + 180 + 1$
$= 56$

Minimum value $= 24$, Maximum value $= 56$

- LOCAL MAXIMA AND MINIMA

* FIRST ORDER DERIVATIVE TEST



Local minima $\Rightarrow (c-h, c),\ f'(c) < 0$
$(c, c+h),\ f'(c) > 0$

Local maxima $\Rightarrow (a-h, a),\ f'(a) > 0$
$(a, a+h),\ f'(a) < 0$

* SECOND ORDER DERIVATIVE TEST

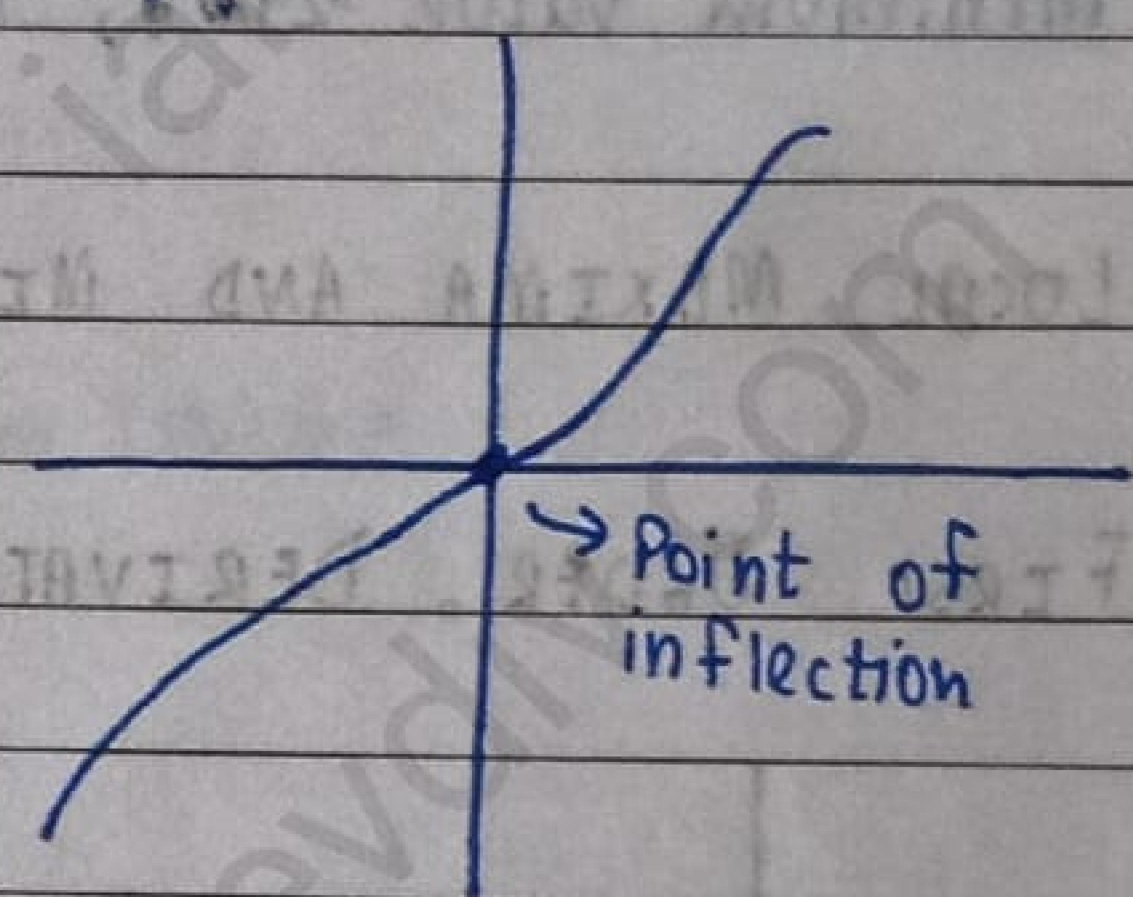
For $f(x)$,
$f'(x) = 0$, $x = a_1, a_2$

$f''(x)]_{x=a_1} < 0$, $f(a_1)$ = maximum value
f is local maxima

$f''(x)]_{x=a_2} > 0$, $f(a_2)$ = minimum value
f is local minima

- POINT OF INFLECTION

When function is differentiable but it has no value of maxima and minima ⇒ Point of inflection



$f(x) = x^3$
$f'(x) = 3x^2$
$f'(x) = 0$
$3x^2 = 0$
$x = 0$

18/8/23

☆ EXERCISE - 6.3

Q1.) Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x-1)^2 + 3$

(i) $f'(x) = 2(2x-1) \times 2 + 0$

$= 4(2x-1)$

$f'(x) = 0$

$\Rightarrow 4(2x-1) = 0$

$\Rightarrow x = \frac{1}{2}$

$f''(x) = 4 \times 2 = 8 > 0$

$x = \frac{1}{2}$ is minimum point

$f\left(\frac{1}{2}\right) = \left(2 \times \frac{1}{2} - 1\right)^2 + 3 = (0)^2 + 3 = 3$

Min. value = 3 Ans

(ii) $f(x) = 9x^2 + 12x + 2$

(ii) $f'(x) = 18x + 12$

$f'(x) = 0$

$\Rightarrow 18x + 12 = 0$

$\Rightarrow 6(3x+2) = 0$

$\Rightarrow x = -\frac{2}{3}$

$f''(x) = 18 > 0$

$x = -\frac{2}{3}$ is minimum point

$f\left[\frac{-2}{3}\right] = \cancel{9} \times \frac{4}{\cancel{9}} + \cancel{12}^{4} \times \frac{-2}{\cancel{3}} + 2$

$= 4 - 8 + 2$

$= -2$

Min. value $= -2$ Ans

(iii) $f(x) = -(x-1)^2 + 10$

(iii) $f'(x) = -[2(x-1)] + 0$

$= -2(x-1)$

$f'(x) = 0$

$\Rightarrow -2(x-1) = 0$

$\Rightarrow x = 1$

$f''(x) = -2 < 0$

$x = 1$ is maximum point

$f(1) = -(1-1)^2 + 10 = 10$

Max. value $= 10$ Ans

(iv) $g(x) = x^3 + 1$

(iv) $g'(x) = 3x^2$

$g'(x) = 0$

$\Rightarrow 3x^2 = 0$

$\Rightarrow x = 0$

$g''(x) = 6x = 6 \times 0 = 0$

Neither maximum nor minimum value exists.

Q2.) Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x+2| - 1$

(i) $|x+2| \geq 0$

$|x+2| - 1 \geq -1$

$f(x) \geq -1$

Range of $f(x) = [-1, \infty)$

Minimum value $= -1$, No maximum value

(ii) $g(x) = -|x+1| + 3$

(ii) $|x+1| \geq 0$

$-|x+1| \leq 0$

$-|x+1| + 3 \leq 3$

$g(x) \leq 3$

Range of $g(x) = (\infty, 3]$

Maximum value $= 3$, No minimum value

(iii) $h(x) = \sin(2x) + 5$

(iii) $-1 \leq \sin x \leq 1$

$-1 \leq \sin 2x \leq 1$

$4 \leq \sin 2x + 5 \leq 6$

$4 \leq h(x) \leq 6$

Minimum value $= 4$, Maximum value $= 6$

(iv) $f(x) = |\sin 4x + 3|$

(iv) $-1 \leq \sin x \leq 1$

$-1 \leq \sin 4x \leq 1$

$2 \leq \sin 4x + 3 \leq 4$

$2 \leq |\sin 4x + 3| \leq 4$

$2 \leq f(x) \leq 4$

Minimum value = 2 , maximum value = 4

(v) $h(x) = x+1, \; x \in (-1,1)$

(v) ~~$-1 < x < 1$~~

~~$0 < x+1 < 2$~~

$h'(x) = 1$

$h'(x) = 0$

$\Rightarrow 1 \neq 0$

Neither minimum nor maximum values exist.

Q3) Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$

(i) $f'(x) = 2x$

$f'(x) = 0$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

$f''(x) = 2 > 0$

$x = 0$ is minimum point

$f(0) = 0^2 = 0$

Local Min. value = 0 Ans

(ii) $g(x) = x^3 - 3x$

(ii) $g'(x) = 3x^2 - 3 = 3(x^2-1)$

$g'(x) = 0$

$\Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow x = \pm 1$

$g''(x) = 6x$

at $x = 1$ ~~(minimum point)~~

$g''(x) = 6 > 0$

(minimum point)

at $x = -1$ ~~(maximum point)~~

$g''(x) = -6 < 0$

(maximum point)

Local maximum $\Rightarrow g(-1) = (-1)^3 - 3 \times -1 = 2$

Local minimum $\Rightarrow g(1) = (1)^3 - 3 \times 1 = -2$

(iii) $h(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

(iii) $h'(x) = \cos x - \sin x$

$h'(x) = 0$

$\Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow \frac{\sin x}{\cos x} = 1$

$\Rightarrow \tan x = \tan \frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4}$

$h''(x) = -\sin x - \cos x$

$= -(\sin x + \cos x)$

$= -\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right)$

$= -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) < 0$

$x = \frac{\pi}{4}$ is maximum point

Local maximum $\cancel{value} \Rightarrow h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2}$ Ans

(iv) $f(x) = \sin x - \cos x, \ 0 < x < 2\pi$

(iv) $f'(x) = \cos x + \sin x$

$f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0$

$\Rightarrow 1 = \frac{-\sin x}{\cos x}$

$\Rightarrow \tan x = -1$

$\Rightarrow x = -\frac{\pi}{4}$

$x = \pi - \frac{\pi}{4}, \ 2\pi - \frac{\pi}{4}$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f''(x) = -\sin x + \cos x$

at $x = \frac{3\pi}{4}$

$f''(x) = -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} < 0$

(maximum point)

at $x = \frac{7\pi}{4}$

$f''(x) = -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} > 0$

(minimum point)

Local minimum $\Rightarrow f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ Ans (a)

Local maximum $\Rightarrow f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$ Ans (b)

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

(v) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$f'(x) = 0$

$\Rightarrow 3(x^2 - 3x - x + 3) = 0$

$\Rightarrow 3[x(x-3) - 1(x-3)] = 0$

$\Rightarrow 3(x-1)(x-3) = 0$

$x = 1, 3$

$f''(x) = 6x - 12$

at $x = 1$

$f''(x) = 6 - 12$

$= -6 < 0$

(maximum point)

at $x = 3$

$f''(x) = 18 - 12$

$= 6 > 0$

(Minimum point)

Local maxima $\Rightarrow f(1) = 1^3 - 6(1)^2 + 9\times 1 + 15$
$= 1 - 6 + 9 + 15$
$= 19$ Ans (a)

Local minima $\Rightarrow f(3) = 3^3 - 6(3)^2 + 9\times 3 + 15$
$= \cancel{27} - \cancel{54} + \cancel{27} + 15$
$= 15$ Ans(b)

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, \quad x > 0$

(ii) $g'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} = \frac{(x+2)(x-2)}{2x^2}$

$g'(x) = 0$
$\Rightarrow \frac{(x+2)(x-2)}{2x^2} = 0$
$x = 2, -2$
But $x > 0 \therefore x = -2$ rejected

$g''(x) = \frac{2x^2 \cdot 2x - (x^2-4)4x}{(2x^2)^2}$
$= \frac{\cancel{4x^3} - \cancel{4x^3} + \cancel{16}^{4} x}{\cancel{4}x^4}$
$= \frac{4}{x^3}$

At $x = 2$
$g''(x) = \frac{4}{2^3} > 0$
(Minimum point)
Local minima,
$g(2) = 2/2 + 2/2 = 2$ Ans

(vii) $g(x) = \frac{1}{x^2+2}$

(vii) $g'(x) = \frac{-2x}{(x^2+2)^2}$

$g'(x) = 0$

$\Rightarrow \frac{-2x}{(x^2+2)^2} = 0$

$\Rightarrow -2x = 0$

$x = 0$

$g''(x) = \frac{-(x^2+2)^2 2 - [-2x \times 2(x^2+2) \times 2x]}{(x^2+2)^4}$

$= \frac{-2(x^2+2)^2 + 8x^2(x^2+2)}{(x^2+2)^4}$

$= \frac{-2(x^2+2)[(x^2+2) - 4x^2]}{(x^2+2)^4}$

$= \frac{-2(x^2+2)(-3x^2+2)}{(x^2+2)^4}$

$= \frac{-2(-3x^2+2)}{(x^2+2)^3}$

At $x=0$

$g''(x) = \frac{-2.(-3 \times 0^2 + 2)}{(0^2+2)^3} = \frac{-4}{8} = \frac{-1}{2} < 0$

(Maxima point)

Local maximum $\Rightarrow$ $\cancel{\frac{1}{(-1/2)^2+2} - \frac{1}{1/4+2}}$ $\frac{1}{0^2+2} = \frac{1}{2}$ Ans

(viii) $f(x) = x\sqrt{1-x}$, $0 < x < 1$

(viii) $\boxed{\cancel{f'(x) = \dfrac{\sqrt{1-x} + x(-1)}{2\sqrt{1-x}} = \dfrac{\sqrt{1-x} - x}{2\sqrt{1-x}}}}$

$$f'(x) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = \frac{2(\sqrt{1-x})^2 - x}{2\sqrt{1-x}} = \frac{2-2x-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$f'(x) = 0$

$\Rightarrow \dfrac{2-3x}{2\sqrt{1-x}} = 0$

$\Rightarrow 2-3x = 0$

$\Rightarrow x = \dfrac{2}{3}$

$$f''(x) = \frac{2\sqrt{1-x} \times -3 + \dfrac{(2-3x)}{\sqrt{1-x}}}{4(1-x)}$$

$$= \frac{-3\sqrt{1-x}}{2(1-x)} + \frac{(2-3x)}{4(1-x)^{3/2}}$$

$$= \frac{-6(1-x) + 2 - 3x}{4(1-x)^{3/2}}$$

$$= \frac{-6 + 6x + 2 - 3x}{4(1-x)^{3/2}}$$

$$= \frac{3x - 4}{4(1-x)^{3/2}}$$

At $x = \dfrac{2}{3}$

$$\Rightarrow f''(x) = \frac{2-4}{4\left(1-\frac{2}{3}\right)^{3/2}} = \frac{-1}{2\left(\frac{1}{3}\right)^{3/2}} < 0$$

(Maximum point)

Local maxima $\Rightarrow f(2/3) = 2/3\sqrt{1-2/3}$

$$= \frac{2 \times \sqrt{1}}{3 \times \sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{9}$$ Ans

Q4) Prove that the following functions do not have maxima or ~~ma~~minima:

(i) $f(x) = e^x$

(i) $f'(x) = e^x$

$f'(x) = 0$

$\Rightarrow e^x = 0$

This is not possible for any value of x

$\therefore f(x)$ does not have maxima or minima

(ii) $g(x) = \log x$

(ii) $g'(x) = \frac{1}{x}$

$g'(x) = 0$

$\Rightarrow \frac{1}{x} = 0$

$\Rightarrow 1 \neq 0$

$\therefore g(x)$ does not have maxima or minima

(iii) $h(x) = x^3 + x^2 + x + 1$

(iii) $h'(x) = 3x^2 + 2x + 1$

$h'(x) = 0$

$\Rightarrow 3x^2 + 2x + 1 = 0$

$ax^2 + bx + c = 0$

$a = 3, b = 2, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 12}}{6}$$

$$= \frac{-2 \pm \sqrt{-8}}{6}$$

x is not real number

$\therefore h(x)$ does not have maxima or minima

Q5.) Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, \; x \in [-2, 2]$

(i) $-2 \le x \le 2$

$\Rightarrow -8 \le x^3 \le 8$

Absolute minimum value $= -8$
Absolute maximum value $= 8$ } Ans

(ii) $f(x) = \sin x + \cos x, \; x \in [0, \pi]$

(ii) $f'(x) = \cos x - \sin x$

$f'(x) = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow x = \frac{\pi}{4}$

$f(0) = \sin 0 + \cos 0 = 1$

$f(\pi/4) = \sin \pi/4 + \cos \pi/4 = \sqrt{2}$

$f(\pi) = \sin \pi + \cos \pi = -1$

Absolute min. value $= -1$, Absolute maximum value $= \sqrt{2}$ Ans

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$

(iii) $f'(x) = 4 - \frac{1}{\cancel{2}} \times \cancel{2}x = 4 - x$

$f'(x) = 0$

$\Rightarrow 4 - x = 0$

$\Rightarrow x = 4$

$f(-2) = 4 \times -2 - \frac{1}{\cancel{2}} \times \cancel{4}^{2} = -8 - 2 = -10$

$f(4) = 4 \times 4 - \frac{1}{\cancel{2}} \times \cancel{16}^{8} = 16 - 8 = 8$

$f(9/2)\ \cancel{4}^{2} \times 9/\cancel{2} - \frac{1}{2} \times \frac{81}{4} = 18 - \frac{81}{8} = \frac{144 - 81}{8} = \frac{63}{8}$

Absolute minimum value $= -10$
Absolute maximum value $= 8$
} Ans

(iv) $f(x) = (x-1)^2 + 3,\ x \in [-3, 1]$

(iv) $-3 \le x \le 1$

$-4 \le x - 1 \le 0$

$0 \le (x-1)^2 \le 16$

$3 \le (x-1)^2 + 3 \le 19$

Absolute minimum value $= 3$
Absolute maximum value $= 19$
} Ans

Q6) Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

Sol. $p'(x) = -72 - 36x$

$p'(x) = 0$

$\Rightarrow -72 = 36x$

$\Rightarrow x = -2$

$p''(x) = -36 < 0$

(Maximum value exists at $x = -2$)

$p(-2) = 41 - 72 \times -2 - 18(4)$

$= 41 + 144 - 72$

$= 185 - 72$

$= 113$ unit Ans

Q7.) Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

Sol. $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$f'(x) = 12x^3 - 24x^2 + 24x - 48$

$= 12(x^3 - 2x^2 + 2x - 4)$

$= 12(x^2(x-2) + 2(x-2))$

$= 12(x^2+2)(x-2)$

$f'(x) = 0$

$\Rightarrow 12(x^2+2)(x-2) = 0$

$x^2 = -2$, $x = 2$

(Not real)

$f(0) = 3(0)^4 - 8(0)^3 + 12(0)^2 - 48(0) + 25 = 25$

$f(2) = 3\times16 - 8\times8 + 12\times4 - 48\times2 + 25 = 48 - 64 + 48 - 96 + 25 = -39$

$f(3) = 3\times81 - 8\times27 + 12\times9 - 48\times3 + 25 = 243 - 216 + 108 - 144 + 25 = 16$

Absolute minimum value $= -39$, Absolute maximum value $= 25$

Q8) At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

Sol. $f(x) = \sin 2x$

$f'(x) = 2\cos 2x$

$f'(x) = 0$

$\Rightarrow 2\cos 2x = 0$

$\Rightarrow \cos 2x = \cos\frac{\pi}{2}$

$\Rightarrow 2x = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}$

$x = \pi + \frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$x = \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{5\pi}{4}$

$f(0) = \sin(2\times 0) = \sin 0 = 0$

$f(\pi/4) = \sin \pi/2 = 1$

$f(3\pi/4) = \sin 3\pi/2 = -1$

$f(5\pi/4) = \sin 5\pi/2 = 1$

$f(7\pi/4) = \sin 7\pi/2 = -1$

$f(2\pi) = \sin 4\pi = 0$

At $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$, $\sin 2x$ attains its maximum value Ans

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Q9) What is the maximum value of the function $\sin x + \cos x$?

Sol. $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f'(x) = 0$

$\Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ Ans

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Q10.) Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

Sol. $f(x) = 2x^3 - 24x + 107$

$f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

$f'(x) = 0$

$\Rightarrow 6(x^2 - 4) = 0$

$\Rightarrow x = \pm 2$

$\cancel{f(-2) = 2\times -8 - 24\times -2 + 107 = -16 + 48 + 107} = 139$

$f(1) = 2 - 24 + 107 = 85$

$f(2) = 2\times 8 - 24\times 2 + 107 = 16 - 48 + 107 = 75$

$f(3) = 2\times 27 - 24\times 3 + 107 = 54 - 72 + 107 = 89$

$f(-3) = 2\times -27 - 24\times -3 + 107 = -54 + 72 + 107 = 125$

$f(-1) = 2\times(-1)^3 - 24\times -1 + 107 = -2 + 24 + 107 = 129$

$f(-2) = 2\times -8 - 24\times -2 + 107 = -16 + 48 + 107 = 139$

For interval $[1, 3] \Rightarrow$ Max. value $= 89$ at $x = 3$
For interval $[-3, -1] \Rightarrow$ Max. value $= 139$ at $x = -2$ } Ans

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Q11.) It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a.

Sol. $f(x) = x^4 - 62x^2 + ax + 9$

$f'(x) = 4x^3 - 124x + a$

$f'(x) = 0$

$\Rightarrow 4x^3 - 124x + a = 0$

$\Rightarrow 4(1)^3 - 124(1) + a = 0$

$\Rightarrow 4 - 124 + a = 0$

$\Rightarrow a = 120$

Q12.) Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

Sol. $f(x) = x + \sin 2x$

$f'(x) = 1 + 2\cos 2x$

$f'(x) = 0$

$1 + 2\cos 2x = 0$

$\Rightarrow 2\cos 2x = -1$

$\Rightarrow \cos 2x = \frac{-1}{2}$

$\Rightarrow \cos 2x = -\cos\frac{\pi}{3}$

$\Rightarrow \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right)$

$\Rightarrow \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow 2x = \frac{2\pi}{3}$

$\Rightarrow x = \frac{\pi}{3}$

$f(0) = 0 + \sin[2(0)] = 0$

$f(\pi/3) = \pi/3 + \sin 2\pi/3 = \pi/3 + \sqrt{3}/2$

$F(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

$$\left.\begin{array}{l}\text{Minimum value} = 0 \\ \text{Maximum value} = 2\pi\end{array}\right\} \underline{\text{Ans}}$$

Q13.) Find two numbers whose sum is 24 and whose product is as large as possible.

Sol. Let numbers be x and y

$x + y = 24$ —— (1)

$P = xy$

$= x(24 - x)$

$P = 24x - x^2$

$\frac{dp}{dx} = 24 - 2x$

$\frac{d^2p}{dx^2} = -2 < 0$ (Local maxima)

$\frac{dp}{dx} = 0$

$\Rightarrow 24 - 2x = 0$

$\Rightarrow x = 12$

$x + y = 24 \Rightarrow 12 + y = 24 \Rightarrow y = 12$

$x = 12,\ y = 12$ Ans

Q14.) Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Sol. $F(x) = xy^3 = (60-y)y^3$

$F'(x) = -y^3 + 3y^2 \boxed{\cancel{(y-60)}} (60-y)$

$= -y^3 - 3y^3 + 180y^2$
$= -4y^3 + 180y^2$
$= 4y^2 \left[\boxed{(y-45)}\right] (45-y)$

$F'(x) = 0$
$\Rightarrow 4y^2(45-y) = 0$
$y \neq 0, y = 45$

$F''(x) = 360y - 12y^2$
at $y = 45$
$F''(x) = 360 \times 45 - 12 \times (45)^2$
$= 16200 - 24300$
$= -8600 < 0$
Product xy^3 is maximum at $y = 45$

$x + y = 60$
$\Rightarrow x + 45 = 60$
$\Rightarrow x = 15$

$x = 15, y = 45$ Ans

Q15.) Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Sol. $x + y = 35$
$P = x^2y^5$
$P = (35-y)^2y^5$

$\frac{dP}{dy} = -2(35-y)y^5 + 5y^4(35-y)^2$

$\frac{dP}{dy} = 0$

$\Rightarrow -2(35-y)\,y^5 + 5y^4(35-y)^2 = 0$

$\Rightarrow (35-y)y^4\,[-2y + 5(35-y)] = 0$

$\Rightarrow (35-y)\,y^4\,(-2y + 175 - 5y) = 0$

$\left.\begin{matrix} y = 0 \\ y = 35 \end{matrix}\right\}$ Rejected

$-7y = -175$

$\Rightarrow y = 25$

$$\frac{d^2P}{dy^2} = -7(35-y)y^4 - y^4(175-7y) + (35-y)(175-7y)\,4y^3$$

at $y = 25$

$$\boxed{\cancel{\frac{d^2P}{dy^2} = -7(35-y)y^4 - y^4(175-7}}$$

$$\frac{d^2P}{dy^2} = -7(35-25)(25)^4 - (25)^4(175-175) + (35-y)(175-175)4y^2$$

$$= -70(25)^4 < 0$$

P has maximum value at $y = 25$

$x + y = 35$

$\Rightarrow x + 25 = 35$

$\Rightarrow x = 10$

$x = 10,\ y = 25$ Ans

(Q16) Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Sol. $x + y = 16$

$f(x) = x^3 + y^3 = x^3 + (16-x)^3$

$f'(x) = 3x^2 - 3(16-x)^2$

$f'(x) = 0$

$\Rightarrow \cancel{3}x^2 = \cancel{3}(16-x)^2$

$\Rightarrow x^2 - 256 - x^2 + 32x = 0$

$\Rightarrow 32x = 256$

$\Rightarrow x = 8$

$f''(x) = 6x + 6 = 6(x+1)$

At $x = 8$

$f''(x) = 54 > 0$

$f(x)$ is minimum at $x = 8$

$x + y = 16$

$\Rightarrow 8 + y = 16$

$\Rightarrow y = 8$

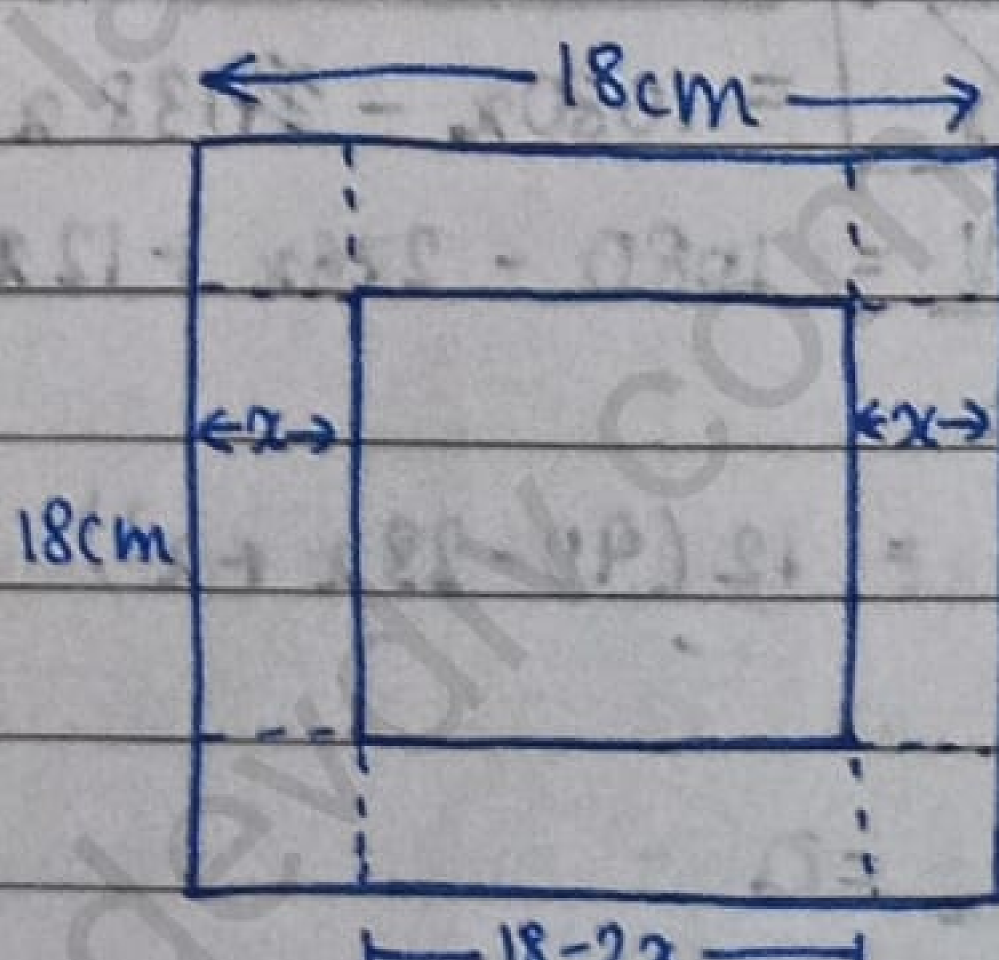
$x = 8,\ y = 8$ Ans

Q17.) A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

Sol. $l = 18 - 2x$

$b = 18 - 2x$

$h = x$



$V = lbh$

$= (18-2x)^2 x$

$\frac{dV}{dx} = (18-2x)^2 + x \times 2(18-2x) \times -2$

$= (18-2x)^2 - 4x(18-2x)$

$\frac{dV}{dx} = 0$

$\Rightarrow (18-2x)^2 - 4x(18-2x) = 0$

$\Rightarrow (18-2x)[18-2x-4x] = 0$

$\Rightarrow (18-2x)(18-6x) = 0$

$x = 3, \quad x = 9$ (rejected)

$\frac{d^2V}{dx^2} = (18-2x)(-6) + (18-6x) \times -2$

$\left(\frac{d^2V}{dx}\right)_{at\, x=3} = -6(18-6) = -72 < 0$

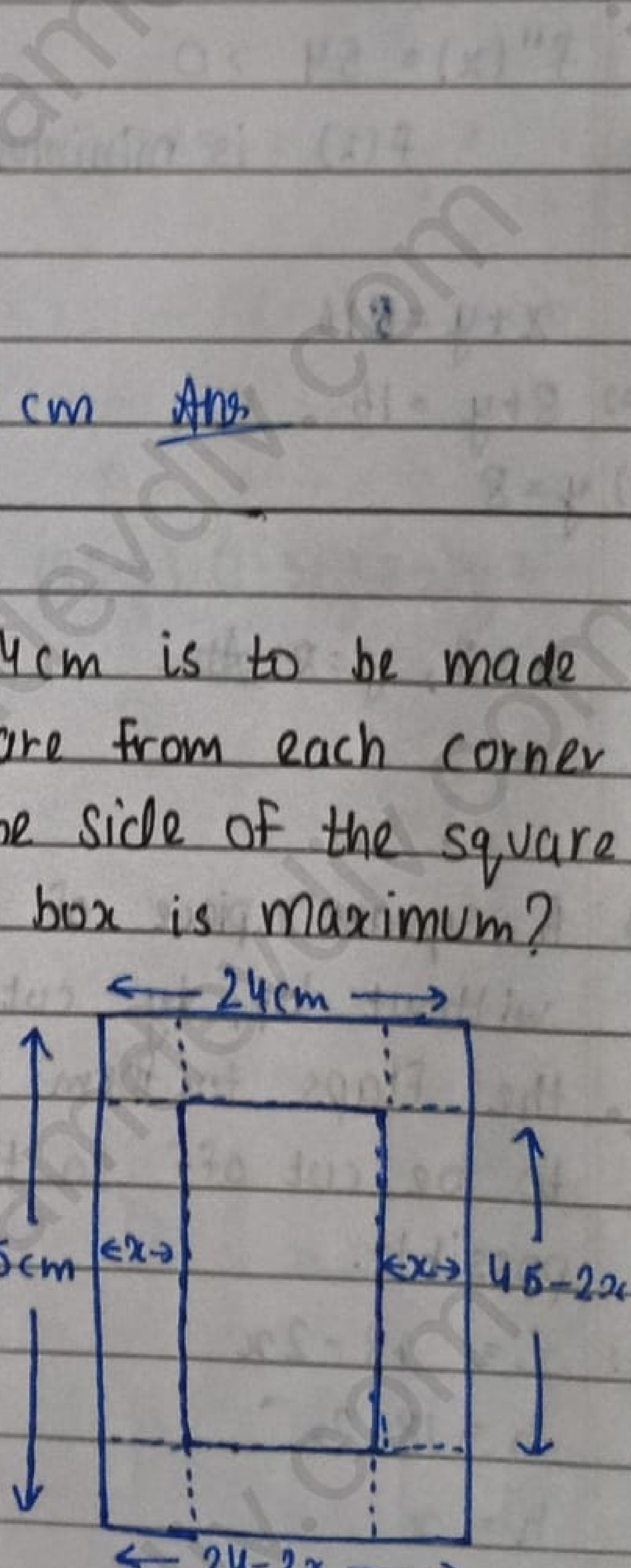
Volume is maximum at $x = 3$ cm Ans

Q18) A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

Sol. $V = lbh$

$= (45-2x)(24-2x)x = (45x-2x^2)(24-2x)$

$= 1080x - 90x^2 - 48x^2 + 4x^3$



$\cancel{\frac{dV}{dx} = (45x-2x^2)(24-2x)}$

$= 1080x - 138x^2 + 4x^3$

$\frac{dV}{dx} = 1080 - 276x + 12x^2$

$= 12(90 - 23x + x^2)$

$\frac{dV}{dx} = 0$

$\Rightarrow 12(90-23x+x^2) = 0$

$\Rightarrow 12(90-18x-5x+x^2) = 0$

$\Rightarrow 12(x-18)(x-5) = 0$

$x = 18$ (rejected) , $x = 5$

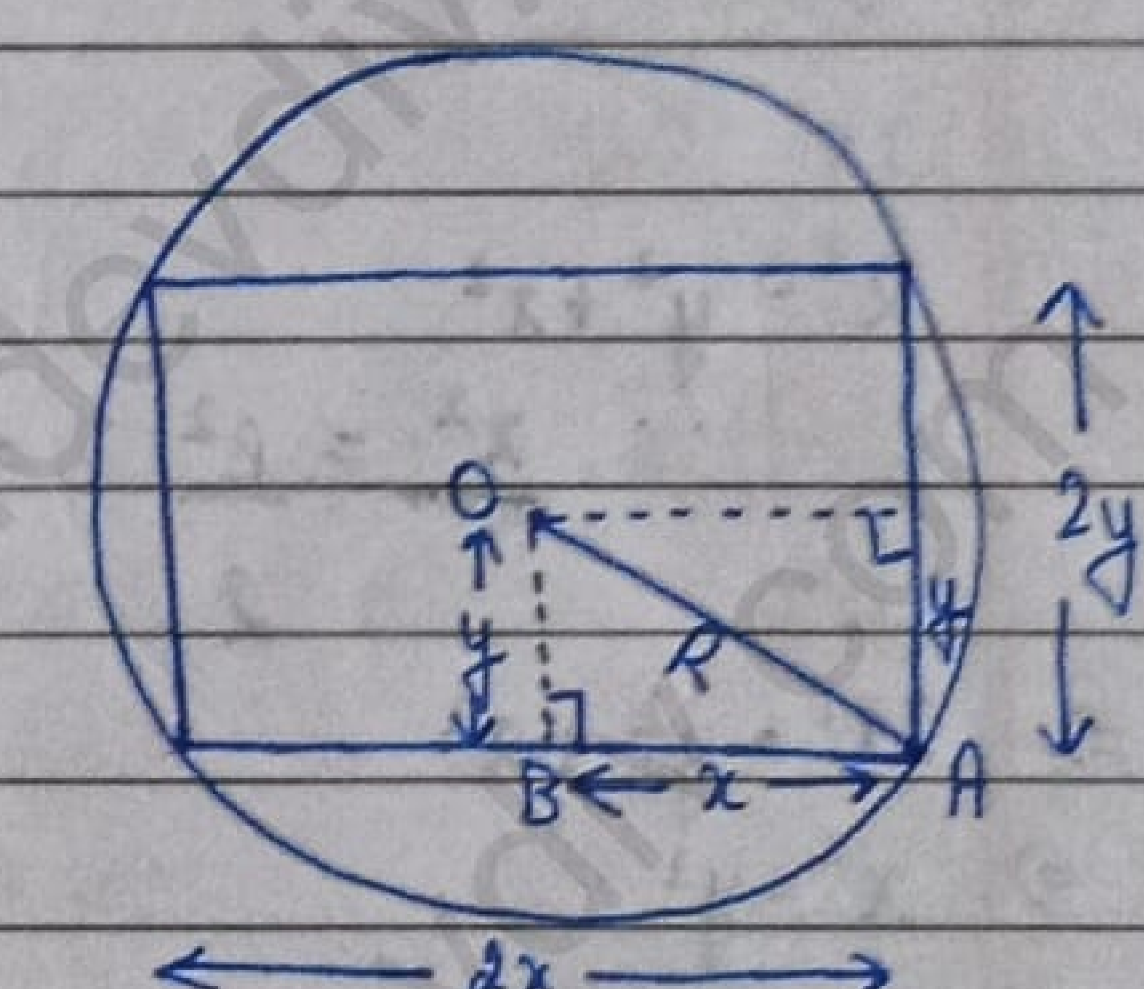
$$\frac{d^2V}{dx^2} = -276 + 24x$$

$$\left(\frac{d^2V}{dx^2}\right)_{at\ x=5} = -276 + 120 = -156 < 0$$

Volume will be maximum at $x = 5$ cm

Q19) Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Sol. Let length of rectangle $= 2x$
Let breadth of rectangle $= 2y$
Radius of circle $= R$



$A = lb$
$= 2x \times 2y$
$= 4xy$

$(OA)^2 = (OB)^2 + (AB)^2$
$\Rightarrow R^2 = y^2 + x^2$
$\Rightarrow y^2 = R^2 - x^2$

$A^2 = 16x^2y^2 \longrightarrow$ Let this be z
$\Rightarrow z = 16x^2(R^2 - x^2) = 16(R^2x^2 - x^4)$
$\Rightarrow \frac{dz}{dx} = 16(R^2 \times 2x - 4x^3)$

$\frac{dz}{dx} = 0$
$\Rightarrow 16(R^2 \times 2x - 4x^3) = 0$

$\Rightarrow 2x(R^2 - 2x^2) = 0$

$x = 0$ (Rejected)

$x^2 = \dfrac{R^2}{2}$

$$\frac{d^2z}{dx^2} = 16(2R^2 - 12x^2)$$

$$\left(\frac{d^2z}{dx^2}\right)_{at\ x = \frac{R}{\sqrt{2}}} = 16\left(2R^2 - \cancel{12}^{6} \times \frac{R^2}{\cancel{2}}\right)$$

$$= -64R^2 < 0$$

Area is maximum at $x = \dfrac{R}{\sqrt{2}}$

$R^2 = y^2 + x^2$

$\because x^2 = \dfrac{R^2}{2} \Rightarrow R^2 = 2x^2$

$\Rightarrow 2x^2 = y^2 + x^2$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$

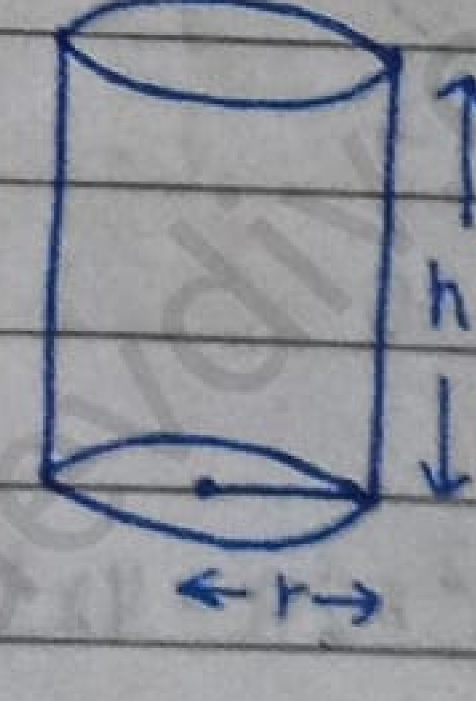
Length and breadth are equal $\Rightarrow$ Square

$\therefore$ Square has the maximum area inscribed in a circle

Hence, proved

Q20.) Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Sol. Let S be surface area, r be radius and h be height of the cylinder



$S = 2\pi r(r + h)$

$\Rightarrow S = 2\pi r^2 + 2\pi rh$

$\Rightarrow S - 2\pi r^2 = 2\pi r h$

$\Rightarrow h = \dfrac{S - 2\pi r^2}{2\pi r}$

$V = \pi r^2 h = \cancel{\pi} r^{\cancel{2}} \times \dfrac{S - 2\pi r^2}{2\cancel{\pi}\cancel{r}} = \dfrac{1}{2}(Sr - 2\pi r^3)$

$\dfrac{dV}{dr} = \dfrac{1}{2}(S - 6\pi r^2)$

$\dfrac{dV}{dr} = 0$

$\Rightarrow \dfrac{1}{2}(S - 6\pi r^2) = 0$

$S - 6\pi r^2 = 0$

$\Rightarrow S = 6\pi r^2$

$\dfrac{d^2V}{dr^2} = \dfrac{1}{2} \times -12\pi r = -6\pi r < 0$

Volume is maximum at $S = 6\pi r^2$

$h = \dfrac{S - 2\pi r^2}{2\pi r} = \dfrac{6\pi r^2 - 2\pi r^2}{2\pi r} = \dfrac{\overset{2}{\cancel{4}}\cancel{\pi} r^{\cancel{2}}}{\cancel{2}\cancel{\pi}\cancel{r}} = 2r$

$\Rightarrow h = 2r$

Hence, proved

Q21) Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Sol. $V = 100 \text{ cm}^3$

$\Rightarrow \pi r^2 h = 100$

$\Rightarrow h = \dfrac{100}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\cancel{\pi} \cancel{r} \times \frac{100}{\cancel{\pi} r^2}$

$= 2\pi r^2 + \frac{200}{r}$

$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2}$

$\frac{dS}{dr} = 0$

$\Rightarrow 4\pi r - \frac{200}{r^2} = 0$

$\Rightarrow \cancel{4}\pi r^3 = \cancel{200}\ 50$

$\Rightarrow r = \left(\frac{50}{\pi}\right)^{1/3}$

$\frac{d^2S}{dr^2} = 4\pi + \frac{2 \times 200}{r^3}$

$= 4\pi + \frac{\cancel{400}^{8}}{\cancel{50}} \times \pi$

$= 12\pi > 0$

Surface area is minimum at $r = \left(\frac{50}{\pi}\right)^{1/3}$

$h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \frac{2 \times 50}{\pi \times \left(\frac{50}{\pi}\right)^{2/3}} = 2 \times \left(\frac{50}{\pi}\right)^{1-\frac{2}{3}} = 2\left(\frac{50}{\pi}\right)^{1/3}$

$\left.\begin{array}{l} \text{Radius} = \left(\frac{50}{\pi}\right)^{1/3} \text{ cm} \\ \text{height} = 2\left(\frac{50}{\pi}\right)^{1/3} \text{ cm} \end{array}\right\}$ Ans

Q22.) A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the the circle is minimum?

Sol. Let length of wire for square be 'x' and that of circle be '$28-x$'

Perimeter of square = 4 x side

$\Rightarrow 4a = x$

$\Rightarrow a = \frac{x}{4}$

Perimeter of circle = $2\pi r$

$\Rightarrow 2\pi r = 28-x$

$\Rightarrow r = \frac{28-x}{2\pi}$

Combined area, A = area of sq. + area of circle

$\Rightarrow A = a^2 + \pi r^2$

$\Rightarrow A = \frac{x^2}{16} + \pi\left(\frac{28-x}{2\pi}\right)^2$

$\Rightarrow A = \frac{x^2}{16} + \frac{\cancel{\pi}(28-x)^2}{4\pi^{\cancel{2}}}$

$\frac{dA}{dx} = \frac{\cancel{2}x}{\cancel{16}8} + \frac{1}{\cancel{4}2\pi} \times \cancel{2}(28-x)(-1)$

$\Rightarrow \frac{dA}{dx} = \frac{x}{8} - \frac{1}{2\pi}(28-x)$

$\frac{dA}{dx} = 0$

$\Rightarrow \frac{x}{8} = \frac{1}{2\pi}(28-x)$

$\Rightarrow \pi x = 112 - 4x$

$\Rightarrow 4x + \pi x = 112$

$\Rightarrow x(4+\pi) = 112$

$\Rightarrow x = \dfrac{112}{4+\pi}$

$$\frac{d^2A}{dx^2} = \frac{1}{8} - \frac{1}{2\pi}(-1)$$

$$= \frac{1}{8} + \frac{1}{2\pi} > 0$$

Combined area is minimum at $x = \dfrac{112}{4+\pi}$

$$28 - x \Rightarrow 28 - \frac{112}{4+\pi} = \frac{112 + 28\pi - 112}{4+\pi} = \frac{28\pi}{4+\pi}$$

Length of the two pieces $= \dfrac{112}{4+\pi}$ m and $\dfrac{28\pi}{4+\pi}$ Ans

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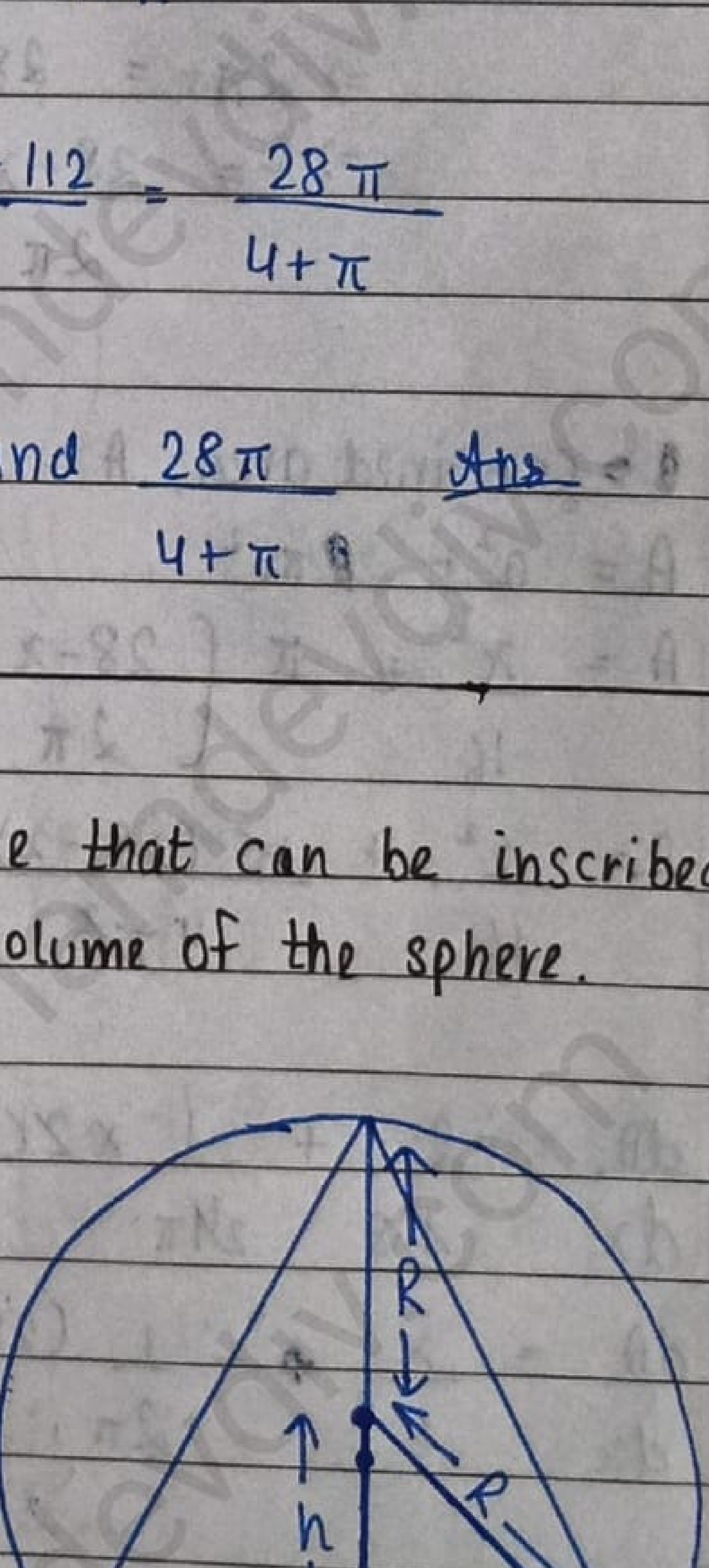
Q23) Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.

SOL. Volume of cone $= \dfrac{1}{3}\pi r^2 (h+R)$



Radius of circle $= R$

$R^2 = h^2 + r^2 \Rightarrow r = \sqrt{R^2 - h^2}$

$\Rightarrow \boxed{\cancel{R = \sqrt{h^2+r^2}}}$

$V = \dfrac{1}{3}\pi(R^2 - h^2)(h+R)$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3}\left[(R^2 - h^2) + (h+R)(-2h)\right]$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3}\left(R^2 - h^2 - 2Rh - 2h^2\right)$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3}\left(-3h^2 - 2Rh + R^2\right)$$

$$\frac{dV}{dh} = 0$$

$$\Rightarrow \frac{\pi}{3}\left(-3h^2 - 2Rh + R^2\right) = 0$$

$$\Rightarrow 3h^2 + 2Rh - R^2 = 0$$

$$\Rightarrow 3h^2 + 3Rh - Rh - R^2 = 0$$

$$\Rightarrow 3h(h+R) - R(h+R) = 0$$

$$\Rightarrow (3h-R)(h+R) = 0$$

$h = -R$ (rejected)

$3h = R \Rightarrow h = \frac{R}{3}$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h - 2R) < 0$$

Volume is maximum at $h = \frac{R}{3}$

Volume of cone $= \frac{1}{3}\pi(R^2 - h^2)(R + h)$

$$= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$$

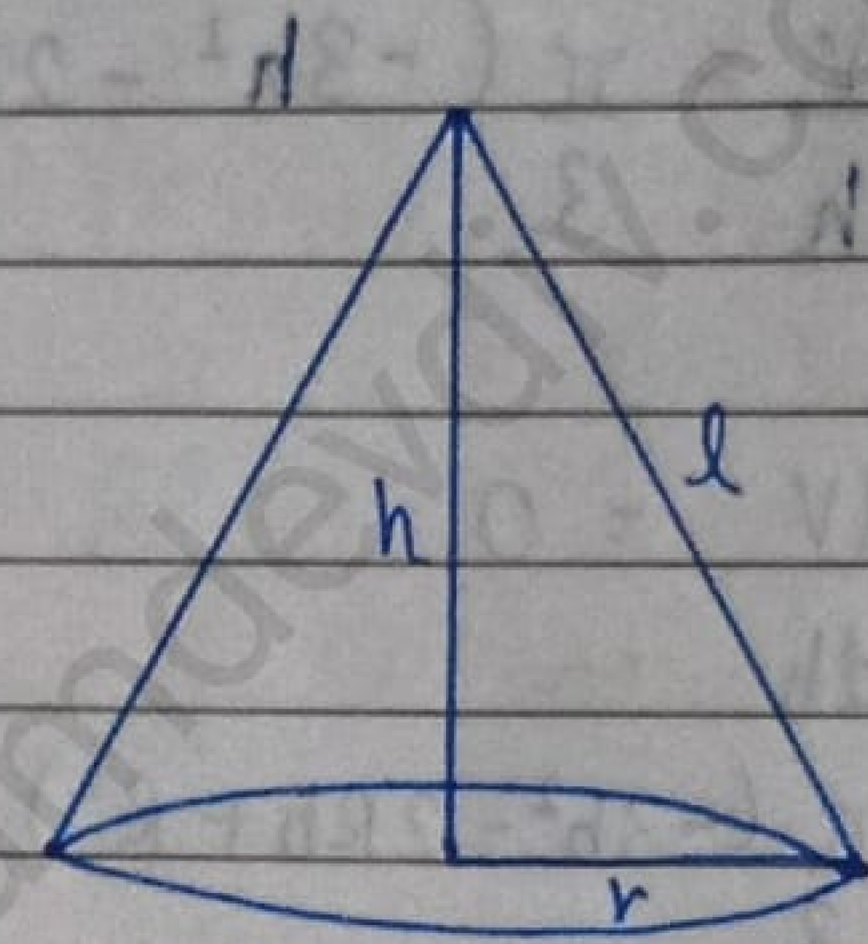
$$= \frac{\pi}{3}\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right)$$

$$= \frac{8}{27}\left(\frac{\pi R^2 \cdot 4R}{3}\right)$$

$$= \frac{8}{27}\left(\frac{4\pi R^3}{3}\right)$$

$= 8/27 \times$ volume of sphere hence, proved

Q24.) Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.



Sol. Volume of cone $= \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \frac{3V}{\pi} = r^2 h$

Let $\frac{3V}{\pi} = k$

$\Rightarrow k = r^2 h$

$\Rightarrow h = \frac{k}{r^2}$

$CSA = \pi r l$

$\Rightarrow S = \pi r \sqrt{h^2 + r^2}$

$\Rightarrow S = \pi r \sqrt{\frac{k^2}{r^4} + r^2}$

$\Rightarrow S = \pi r \sqrt{\frac{k^2 + r^6}{r^4}}$

$\Rightarrow S^2 = \pi^2 r^2 \frac{(k^2 + r^6)}{r^4}$

$\Rightarrow S^2 = \frac{\pi^2}{r^2}(k^2 + r^6)$

$\Rightarrow f(r) = \frac{\pi^2 k^2}{r^2} + \pi^2 r^4$

$f'(r) = \frac{-2\pi^2 k^2}{r^3} + 4\pi^2 r^3$

$f'(r) = 0$

$\Rightarrow \frac{-2\pi^2 k^2}{r^3} + 4\pi^2 r^3 = 0$

$\Rightarrow \cancel{4}^{2}\pi^2 r^3 = \dfrac{\cancel{2}\pi^2 k^2}{r^3}$

$\Rightarrow k^2 = 2r^6$

$f''(r) = \dfrac{6\pi^2 k^2}{r^4} + 12\pi^2 r^2 > 0$

CSA is minimum

$k = hr^2$

$\Rightarrow k^2 = h^2 r^4$

$\Rightarrow 2r^6 = h^2 r^4$

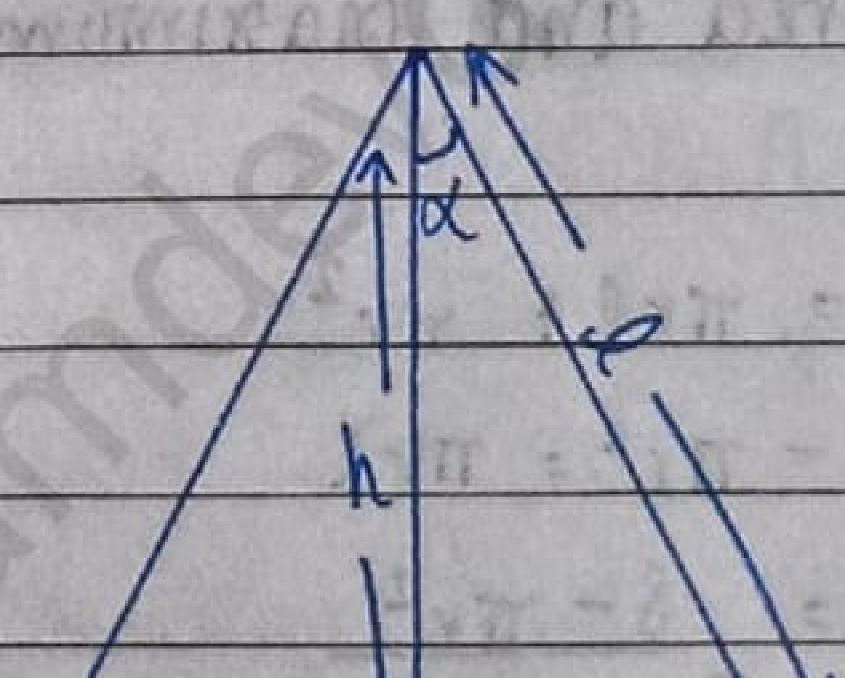
$\Rightarrow 2r^2 = h^2$

$\Rightarrow h = \sqrt{2}\, r$

hence, proved

Q25.) Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Sol. $V = \dfrac{1}{3}\pi r^2 h$



$\Rightarrow V = \dfrac{1}{3}\pi(\ell^2 - h^2)h$

$\Rightarrow V = \dfrac{1}{3}\pi(\ell^2 h - h^3)$

$\dfrac{dV}{dh} = \dfrac{1}{3}\pi(\ell^2 - 3h^2)$

$\dfrac{dV}{dh} = 0$

$\Rightarrow \dfrac{1}{3}\pi(\ell^2 - 3h^2) = 0$

$\Rightarrow \ell^2 = 3h^2$

$$\Rightarrow h^2 = \frac{l^2}{3}$$

$$\Rightarrow h = \frac{l}{\sqrt{3}}$$

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(0-6h) < 0$$

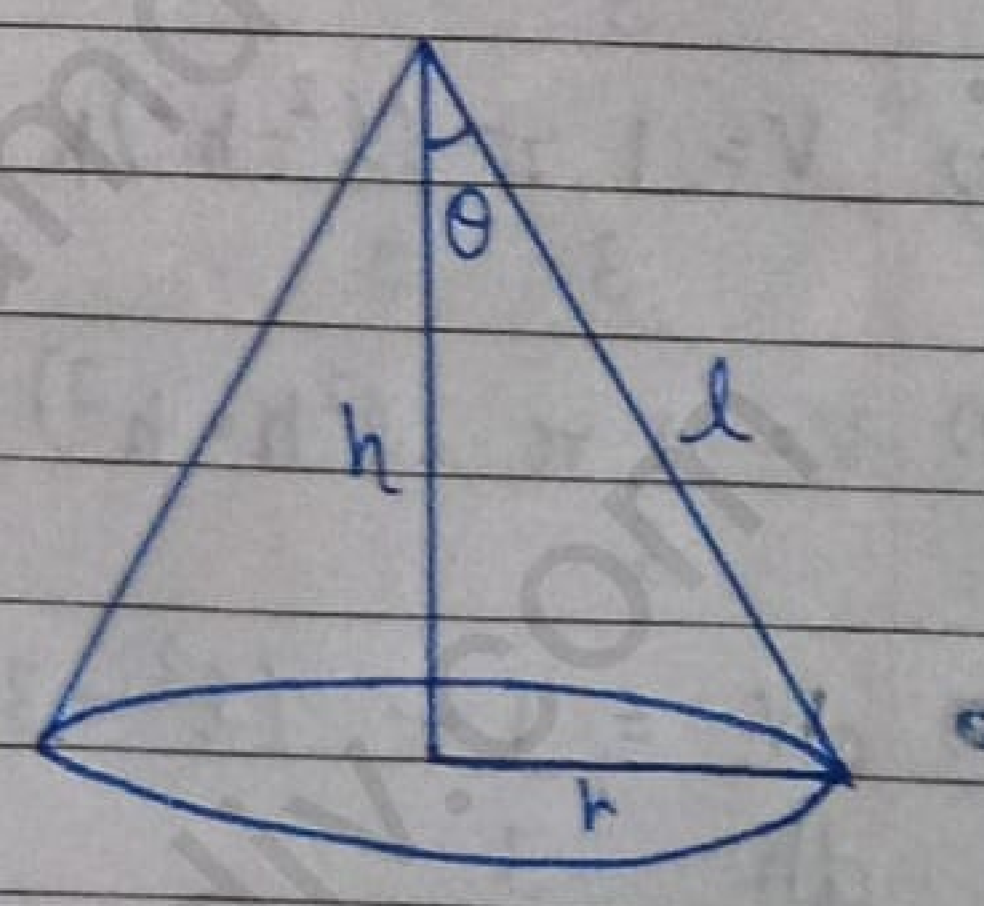
Volume is maximum

$$\tan\alpha = \frac{r}{h} = \frac{\frac{\sqrt{2}}{\sqrt{3}} \cancel{l}}{\frac{\cancel{l}}{\sqrt{3}}} = \sqrt{2}$$

$$\Rightarrow \alpha = \tan^{-1}\sqrt{2}$$

Hence, proved

Q26) Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$



Sol. $S = \pi r l + \pi r^2$

$$\Rightarrow S - \pi r^2 = \pi r l$$

$$\Rightarrow l = \frac{S - \pi r^2}{\pi r}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 \sqrt{\frac{(S-\pi r^2)^2}{\pi^2 r^2} - r^2}$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 \sqrt{\frac{(S-\pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}}$$

$\Rightarrow V = \frac{r}{3}\sqrt{(S-\pi \cdot r^2 - \pi r^2)(S - \cancel{\pi r^2} + \cancel{\pi r^2})}$

$\Rightarrow V = \frac{r}{3}\sqrt{(S-2\pi r^2)S}$

$\Rightarrow V^2 = \frac{r^2}{9} S(S-2\pi r^2)$

$f(r) = \boxed{\cancel{\frac{r^2 S(S-2\pi r^2)}{9}}} \quad \frac{S^2 r^2}{9} - \frac{2S\pi r^4}{9}$

$\Rightarrow f'(r) = \frac{2S^2 r}{9} - \frac{8S\pi r^3}{9}$

$f'(r) = 0$

$\Rightarrow \frac{2S^2 r - 8S\pi r^3}{9} = 0$

$\Rightarrow \cancel{2}S^{\cancel{2}}\cancel{r} = \cancel{8}^{4}\cancel{S}\pi r^{3-1}$

$\Rightarrow S = 4\pi r^2$

$f''(r) = \boxed{\cancel{\frac{2S^2}{9} - \frac{8S\pi\, 8r^2}{9\,3}}} \quad \frac{2S^2 - 8S\pi 3r^2}{9} = \frac{32\pi^2 r^4 - 96\pi^2 r^4}{9}$

$= \frac{-64\pi^2 r^4}{9} < 0$

Volume is maximum

$S = 4\pi r^2$

$\Rightarrow \pi r l + \pi r^2 = 4\pi r^2$

$\Rightarrow \cancel{\pi}\cancel{r} l = 3\cancel{\pi} r^2$

$\Rightarrow \frac{r}{l} = \frac{1}{3}$

$\sin\theta = \frac{r}{l} = \frac{1}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right)$ Hence, proved

Choose the correct answer in Questions 27 and 29.

Q27.) The point on the curve $x^2 = 2y$ which is nearest to the point $(0,5)$ is

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) $(0,0)$ (D) $(2,2)$

Sol. Let point $P(x, y)$ be nearest to $Q(0,5)$

$PQ^2 = (x-0)^2 + (y-5)^2$

$\Rightarrow PQ^2 = x^2 + y^2 + 25 - 10y$

$\Rightarrow PQ^2 = 2y + y^2 + 25 - 10y$

$\Rightarrow f(y) = y^2 - 8y + 25$

$f'(y) = 2y - 8$

$f'(y) = 0$

$\Rightarrow 2y - 8 = 0$

$\Rightarrow y = 4$

$x^2 = 2y$

$\Rightarrow x^2 = 8$

$\Rightarrow x = 8 = 2\sqrt{2}$

$\therefore$ (A) $(2\sqrt{2}, 4)$ Ans

Q28.) For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

Sol. $f(x) = \frac{1-x+x^2}{1+x+x^2}$

$f'(x) = \frac{(1+x+x^2)(-1+2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$

$= \frac{-1 - \cancel{x} - x^2 + \cancel{2x} + 2x^2 + \cancel{2x^3} - 1 + \cancel{x} - x^2 - \cancel{2x} + 2x^2 - \cancel{2x^3}}{(1+x+x^2)^2}$

$= \frac{2x^2 - 2}{(1+x+x^2)^2}$

$$= \frac{2(x^2-1)}{(1+x+x^2)^2}$$

$$= \frac{2(x+1)(x-1)}{(1+x+x^2)^2}$$

$f'(x)=0$

$x=-1, 1$

at $x=-1$, $f(x)=3$ and at $x=1$, $f(x)=\frac{1}{3}$ which is minimum value

$\therefore$ ~~(B) 1 Ans~~ $\therefore$ (D) $\frac{1}{3}$ Ans

Q29) The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{1/3}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Sol. $f(x) = (x^2-x+1)^{1/3}$

$f'(x) = \frac{1}{3}(x^2-x+1)^{-2/3} \cdot (2x-1) = \frac{(2x-1)}{3(x^2-x+1)^{2/3}}$

$f'(x)=0$

$x = \frac{1}{2}$

$f(0) = [0(0-1)+1]^{1/3} = 1$

$f\left(\frac{1}{2}\right) = \left[\frac{1}{2}\left(\frac{1}{2}-1\right)+1\right]^{1/3} = \left(\frac{-1}{4}+1\right)^{1/3} = \left(\frac{3}{4}\right)^{1/3}$

$f(1) = (1(1-1)+1)^{1/3} = 1$

$\therefore$ (C) 1 Ans

2/8/23

★ ~~Exe~~ MISCELLANEOUS EXERCISE ON CHAPTER 6

Q1.) Show that the function given by $f(x) = \frac{\log x}{x}$ ~~has~~ is maximum at $x=e$.

Sol. $f'(x) = \frac{x \times \frac{1}{x} - \log x}{x^2} = \frac{1-\log x}{x^2}$

$f'(x) = 0$

$\Rightarrow \frac{1-\log x}{x^2} = 0$

$\Rightarrow \log x = 1$

$\Rightarrow x = e$

$f''(x) = \frac{x^2 \times \frac{-1}{x} - (1-\log x)2x}{x^4}$

$= \frac{-x - 2x + 2x\log x}{x^4}$

$= \frac{-3x + 2x\log x}{x^4}$

$= \frac{-x(3-2\log x)}{x^4}$

$= \frac{-(3-2\log x)}{x^3}$

$f''(e) = \frac{-(3-2\log e)}{e^3} = \frac{-(3-2)}{e^3} = \frac{-1}{e^3} < 0$

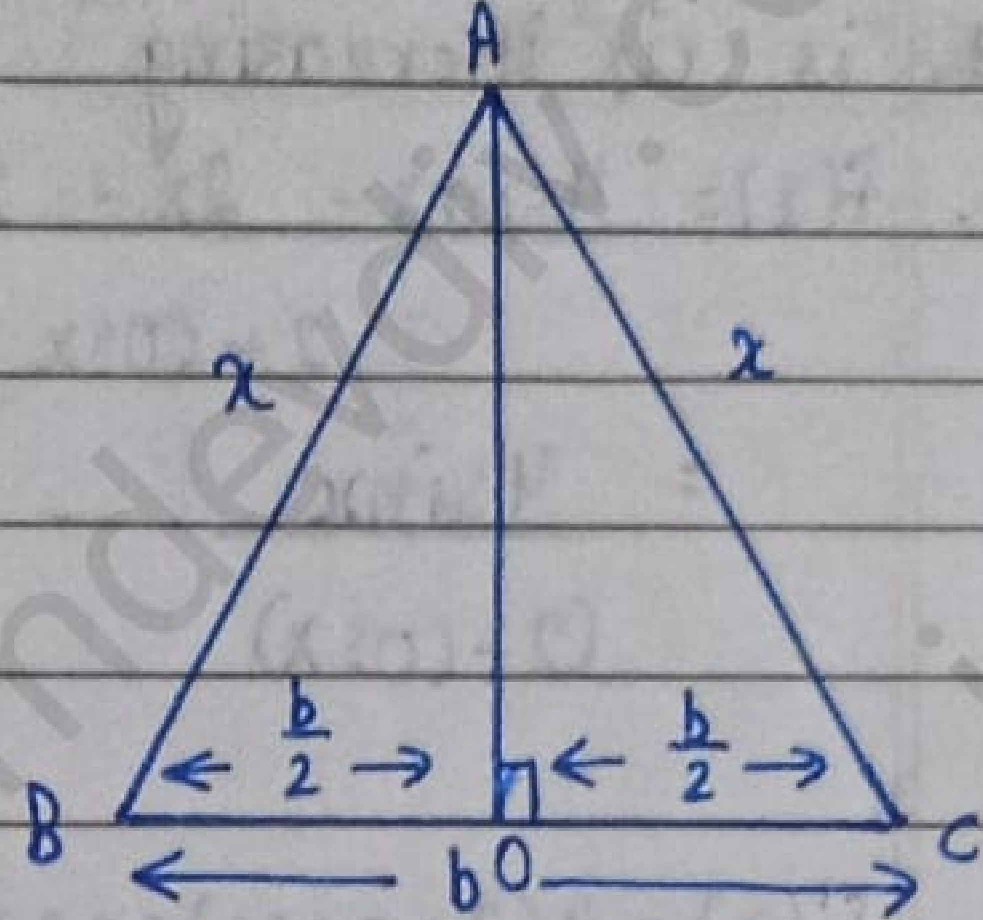
$x=e$ is point of maxima

$\therefore f(x)$ is maximum at $x=e$

Hence, proved

Q2.) The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Sol. $x^2 = \frac{b^2}{4} + (OA)^2$



$\Rightarrow (OA)^2 = x^2 - \frac{b^2}{4}$

$\Rightarrow (OA) = \cancel{\boxed{\frac{4x^2-b^2}{4}}} \sqrt{x^2 - \frac{b^2}{4}}$

$\frac{dx}{dt} = 3 \text{ cm/sec.}$

$A = \frac{1}{2} \times \text{Base} \times \text{height} = \frac{1}{2} \times b \times \sqrt{x^2 - \frac{b^2}{4}}$

$\frac{dA}{dt} = \frac{d}{dt}\left[\frac{b}{2}\sqrt{x^2-\frac{b^2}{4}}\right] \times \frac{\cancel{dt}\,dx}{\cancel{dt}\,dx}$

$= \frac{d}{dx}\left\{\frac{b}{2}\sqrt{x^2-\frac{b^2}{4}}\right\} \frac{dx}{dt}$

$= \frac{b}{2} \times \frac{1}{\cancel{2}\sqrt{x^2-\frac{b^2}{4}}} \times \cancel{2}x \frac{dx}{dt}$

$= \frac{3bx}{2\sqrt{x^2-\frac{b^2}{4}}}$

$\left(\frac{dA}{dt}\right)_{at\ x=b} = \frac{3b^2}{2\sqrt{b^2-\frac{b^2}{4}}} = \frac{3b^2}{2\sqrt{\frac{3b^2}{4}}} = \frac{3b^2}{\frac{\cancel{2}\sqrt{3b^2}}{\cancel{2}}} = \frac{\sqrt{3}\,\cancel{3}\,b^{\cancel{2}}}{\cancel{\sqrt{3}} \times \cancel{b}}$

$= \sqrt{3}\,b \text{ cm}^2/\text{sec}$ Ans

Q3.) Find the intervals in which the function f given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$$

is (i) increasing (ii) decreasing

Sol. $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2+\cos x} = \dfrac{4\sin x - x(2+\cos x)}{(2+\cos x)}$

$= \dfrac{4\sin x}{(2+\cos x)} - x$

$$f'(x) = \frac{4(2+\cos x)\cos x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

$$= \frac{\cancel{8}^{\cos x} + 4\cos^2 x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + \cancel{4} - \cancel{4} - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$= \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

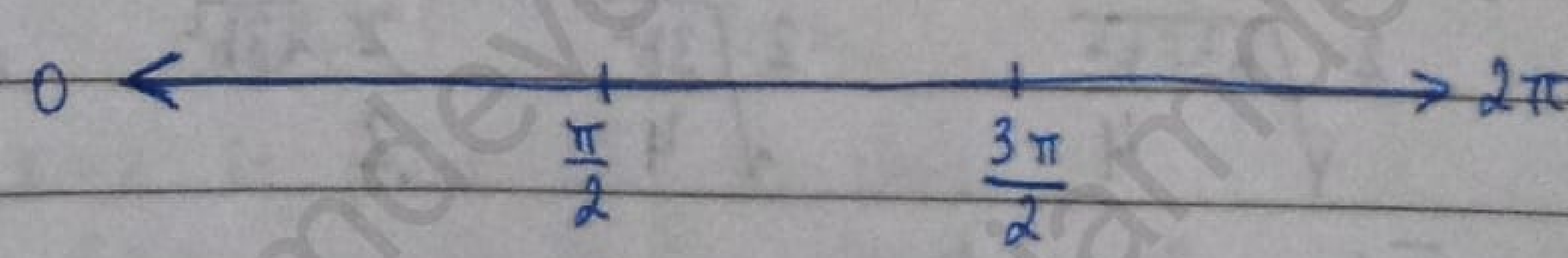
$f'(x) = 0$

$\Rightarrow \dfrac{\cos x(4-\cos x)}{(2+\cos x)^2} = 0$

$\cos x = 0$, $\cos x = 4$ (rejected)

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Interval $= [0, 2\pi]$



$f'(0) = \dfrac{\cos 0(4-\cos 0)}{(2+\cos 0)^2} = \dfrac{4-1}{(2+1)^2} = \dfrac{3}{9} = \dfrac{1}{3}$

$$f'\left(\frac{\pi}{2}\right) = \frac{\cos \pi/2\,(4-\cos \pi/2)}{(2+\cos \pi/2)^2} = 0$$

$$f'\left(\frac{3\pi}{2}\right) = \frac{\cos 3\pi/2\,(4-\cos 3\pi/2)}{(2+\cos 3\pi/2)^2} \cancel{=} \boxed{\cancel{\frac{1/2\,(4-1/2)}{(2+1/2)^2} = \frac{\frac{1}{2}\times\frac{7}{2}}{\frac{5}{2}} = \frac{7\times2}{5\times2\times2}}}$$

$$= 0$$

$$f(2\pi) = \frac{\cos 2\pi\,(4-\cos 2\pi)}{(2+\cos 2\pi)^2} = \frac{1\,(4-1)}{(2+1)^3} = \frac{3}{9} = \frac{1}{3}$$

$x \in \left(0, \frac{\pi}{2}\right)$ $\quad f'(x) > 0$

$x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ $\quad f'(x) < 0$

$x \in \left(\frac{3\pi}{2}, 2\pi\right)$ $\quad f'(x) > 0$

(i) $f(x)$ is increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

(ii) $f(x)$ is decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Q4.) Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing

Sol. $f'(x) = 3x^2 + \frac{3x^2}{x^6} = 3x^2 + \frac{3}{x^4} = 3\left(x^2 - \frac{1}{x^4}\right)$

$f'(x) = 0$

$\Rightarrow 3\left(x^2 - \frac{1}{x^4}\right) = 0$

$\Rightarrow x^2 = \frac{1}{x^4}$

$\Rightarrow x^6 = 1$

$\Rightarrow x = \pm 1$

$-\infty \longleftarrow -1 \quad 1 \longrightarrow \infty$

$x \in (-\infty, -1) \qquad f'(x) > 0$

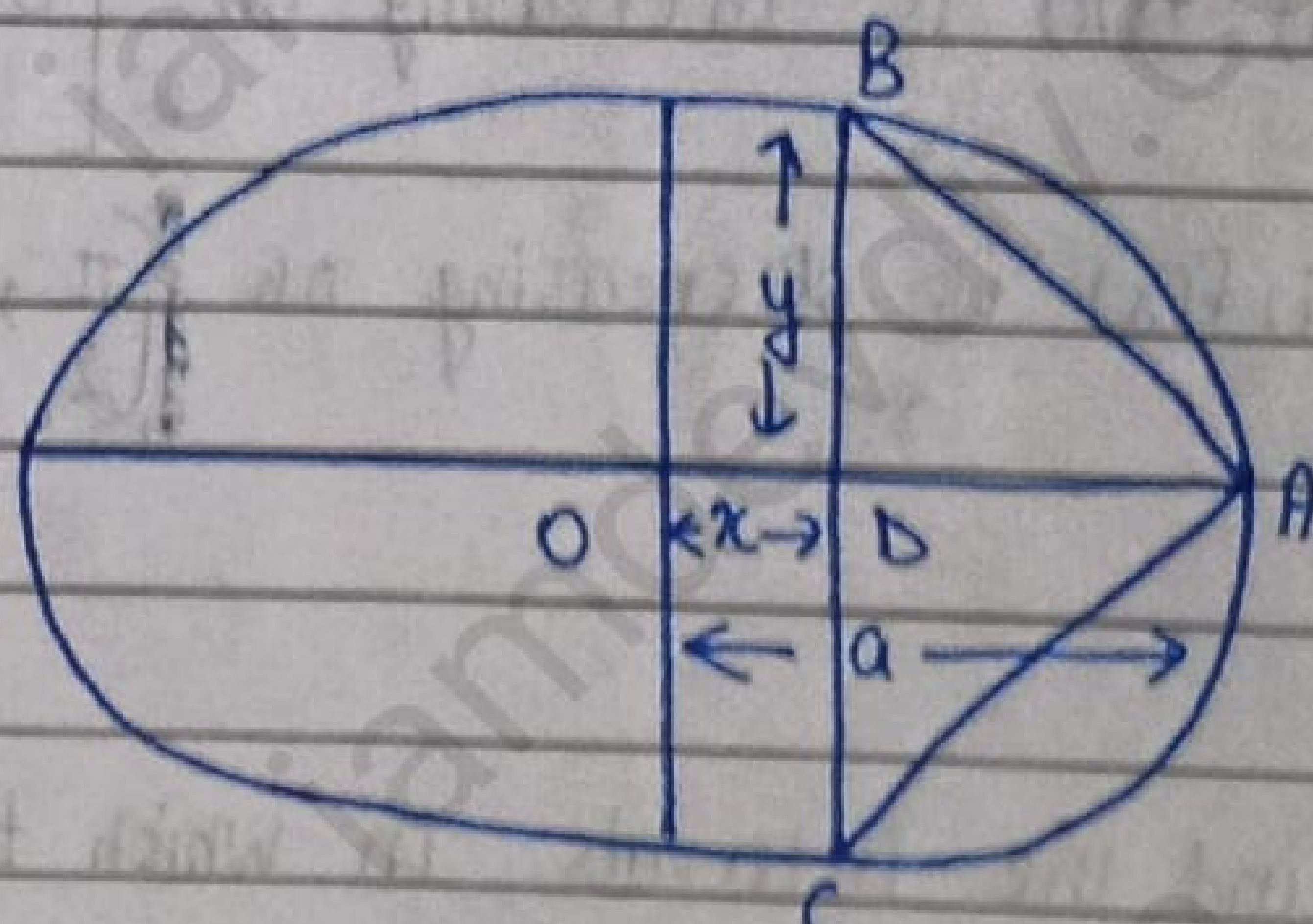
$x \in (-1, 1) \qquad f'(x) < 0$

$x \in (1, \infty) \qquad f'(x) > 0$

(i) $f(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$

(ii) $f(x)$ is decreasing on $(-1, 1)$

Q5.) Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.



Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$

$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$

$OA = a$

$OD = x$

$AD = (a - x)$

$BC = 2y$

$A = \frac{1}{\cancel{2}} \times \cancel{2}y\,(a - x)$

$= y(a-x)$

$= \frac{b}{a}\sqrt{a^2-x^2}\,(a-x)$

$$\frac{dA}{dx} = \frac{b}{a}\left[\frac{1(0-2x)}{2\sqrt{a^2-x^2}}(a-x) + \sqrt{a^2-x^2}\,(-1)\right]$$

$$= \frac{b}{a}\left[\frac{x(x-a)}{\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}\right]$$

$$= \frac{b}{a}\left[\frac{x(x-a)-(a^2-x^2)}{\sqrt{a^2-x^2}}\right]$$

$$= \frac{b}{a}\left[\frac{x^2-ax-a^2+x^2}{\sqrt{a^2-x^2}}\right]$$

$$= \frac{b}{a}\left[\frac{2x^2-ax-a^2}{\sqrt{a^2-x^2}}\right]$$

$$\frac{dA}{dx} = 0$$

$\Rightarrow 2x^2-ax-a^2=0$

$\Rightarrow 2x^2-2ax+ax-a^2=0$

$\Rightarrow 2x(x-a)+a(x-a)=0$

$\Rightarrow (2x+a)(x-a)=0$

$x=a$, (Rejected) (Area = Zero) $\quad x = \frac{-a}{2}$

$$\frac{d^2A}{dx^2} = \frac{b}{a}\left[\frac{\sqrt{a^2-x^2}\,(4x-a) - \frac{(2x^2-ax-a^2)}{2\sqrt{a^2-x^2}} \times -2x}{(a^2-x^2)}\right]$$

$$= \frac{b}{a}\left[\frac{(4x-a)}{\sqrt{a^2-x^2}} + \frac{2x^3-ax^2-a^2x}{(a^2+x^2)^{3/2}}\right] < 0$$

Area is maximum at $x = \frac{-a}{2}$

$$A = \frac{b}{\cancel{a}} \frac{\sqrt{3}\cancel{a}}{2}\left[a + \frac{a}{2}\right]$$

$$= \frac{b}{2}\sqrt{3} \times \frac{3a}{2}$$

$$= \frac{3\sqrt{3}}{4} ab \text{ Ans}$$

Q6.) A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2m and volume is $8m^3$. If building of tank costs Rs 70 per sq. metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

Sol. Let $l = x$, $b = y$

Area of base $= xy$

Area of walls $= 2h(l+b)$

$= 2 \times 2(x+y)$

$= 4(x+y)$

$v = 8m^3$

$\Rightarrow lbh = 8m^3$

$\Rightarrow xy \times 2 = 8$

$\Rightarrow xy = 4$

$$z = 70xy + 45 \times 4(x+y)$$

$$= 70 \times 4 + 180\left[x + \frac{4}{x}\right]$$

$$= 280 + 180\left[x + \frac{4}{x}\right]$$

$$\frac{dz}{dx} = 0 + 180\left[1 - \frac{4}{x^2}\right]$$

$$\frac{d^2z}{dx^2} = 180 \times \frac{8}{x^3}$$

$$= \frac{1440}{x^3} > 0$$

Volume is minimum

$$\frac{dz}{dx} = 0$$

$$\Rightarrow 180\left[1 - \frac{4}{x^2}\right] = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x-2) = 0$$

$x = 2, -2$ (Rejected)

$y = 2$

Cost for the base $= 70 \times xy = 70 \times 4 = 280$ Rs.

Cost for the sides $= 45 \times 4(x+y) = 45 \times 16 = 720$ Rs.

Total cost $=$ ₹1000 Ans

Q7.) The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Sol. $P_c + P_s = k$

Let radius of circle $= r$

Let side of square $= x$

$$2\pi r + 4x = k$$

$$\Rightarrow x = \frac{k - 2\pi r}{4}$$

$$z = \pi r^2 + x^2$$

$\Rightarrow z = \pi r^2 + \left[\frac{k - 2\pi r}{4}\right]^2$

$\Rightarrow \frac{dz}{dr} = 2\pi r + \frac{\cancel{2}(k - 2\pi r)}{\cancel{4}2} \times \frac{-\cancel{2}\pi}{\cancel{4}2}$

$= \frac{8\pi r - \pi k + 2\pi^2 r}{4}$

$= \frac{\pi(8r - k + 2\pi r)}{4}$

$\frac{d^2z}{dr^2} = \frac{\pi(8 + 2\pi)}{4} > 0$

Sum of the areas is minimum

$\frac{dz}{dr} = 0$

$\Rightarrow \frac{\pi(8r - k + 2\pi r)}{4} = 0$

$\Rightarrow 8r - k + 2\pi r = 0$

$\Rightarrow 2r(4 + \pi) = k$

$r = \frac{k}{2(4+\pi)}$

$x = \frac{k - 2\pi \times \frac{k}{2(4+\pi)}}{4} = \frac{k(4 + \pi - \pi)}{4(4+\pi)} = \frac{k}{(4+\pi)}$

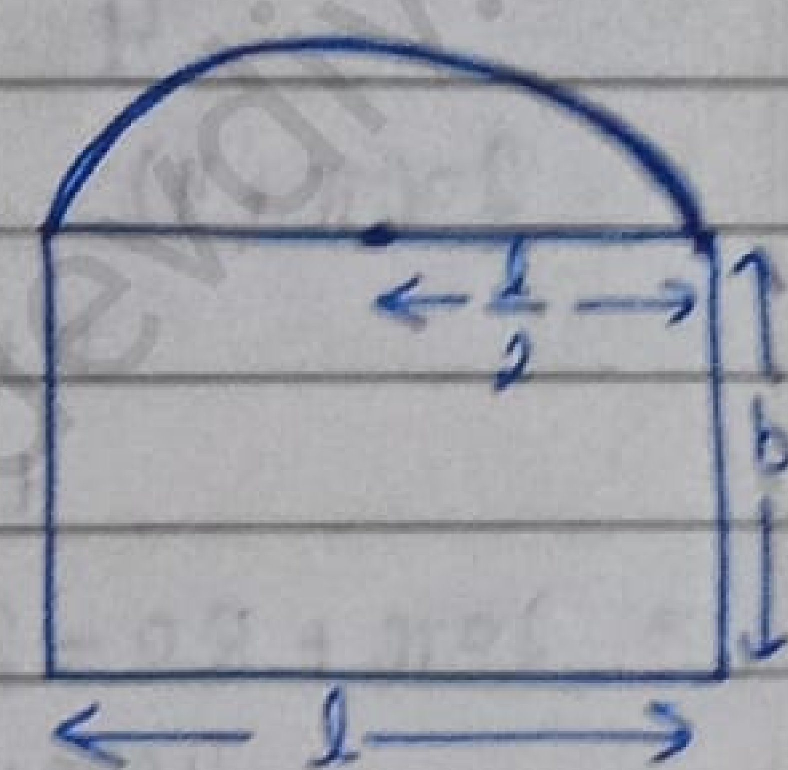
$r = 2x$

$\therefore$ Sum of areas is least when the side of square is double the radius of the circle

Hence, proved

Q8) A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10m. Find the dimensions of the window to admit maximum light through the whole opening.

Sol. $\frac{\pi l}{2} + l + 2b = 10$



$\Rightarrow \pi l + 2l + 4b = 20$

$\Rightarrow \cancel{\pi l + 2l +} 4b = 20 - \pi l - 2l$

$\Rightarrow b = \frac{20 - \pi l - 2l}{4}$

$A = lb + \frac{\pi l^2}{8}$

$\Rightarrow A = \frac{l(20 - \pi l - 2l)}{4} + \frac{\pi l^2}{8}$

$\Rightarrow A = \frac{2l(20 - \pi l - 2l) + \pi l^2}{8}$

$\Rightarrow A = \frac{40l - 2\pi l^2 - 4l^2 + \pi l^2}{8}$

$\Rightarrow A = \frac{40l - \pi l^2 - 4l^2}{8}$

$\frac{dA}{dl} = \frac{1}{8}(40 - 2\pi l - 8l)$

$\frac{dA}{dl} = 0$

$\Rightarrow 40 - 2\pi l - 8l = 0$

$\Rightarrow 2\pi l + 8l = 40$

$\Rightarrow \pi l + 4l = 20$

$\Rightarrow l(\pi + 4) = 20$

$\Rightarrow l = \frac{20}{\pi + 4}$

$$b = \frac{20 - l(\pi+2)}{4}$$

$$= \frac{20 - \frac{20}{\pi+4}(\pi+2)}{4}$$

$$= \frac{\frac{20(\pi+4) - 20(\pi+2)}{\pi+4}}{4}$$

$$= \frac{20\pi + 80 - 20\pi - 40}{4(\pi+4)}$$

$$= \frac{40}{4(\pi+4)}$$

$$= \frac{10}{\pi+4}$$

Length $= \frac{20}{\pi+4}$ m; Breadth $= \frac{10}{\pi+4}$ m ~~Ans~~

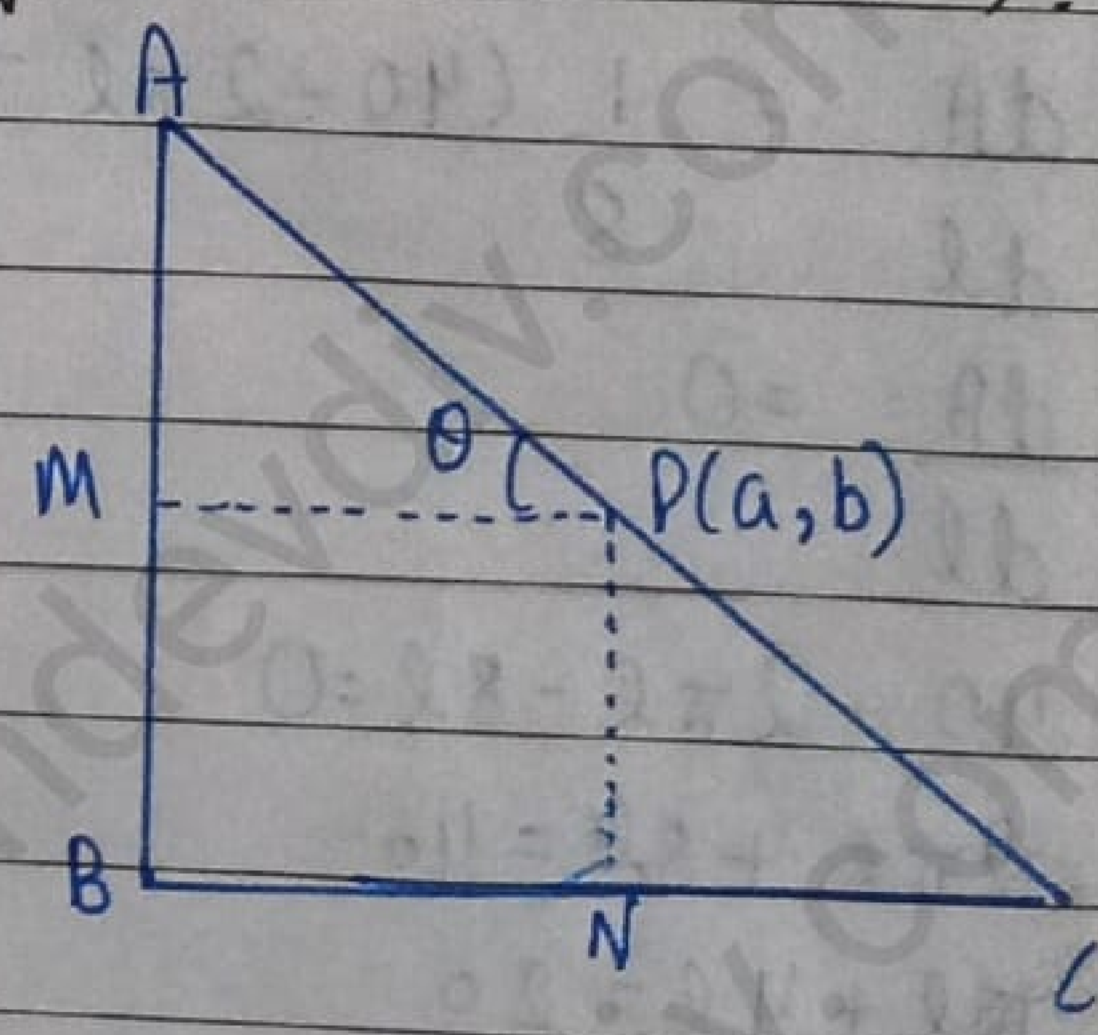
Q9.) A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.
Show that the minimum length of the hypotenuse is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

Sol. In ΔPNC,

$$\sin\theta = \frac{PN}{PC}$$

$$\Rightarrow PC = \frac{PN}{\sin\theta}$$

$$\Rightarrow PC = b\,\mathrm{cosec}\,\theta$$



In ΔAMP,

$$\cos\theta = \frac{MP}{AP}.$$

$\Rightarrow AP = a\sec\theta$

$AC = l = b\,\text{cosec}\,\theta + a\sec\theta$

$\frac{dl}{d\theta} = -b\,\text{cosec}\,\theta\cot\theta + a\sec\theta\tan\theta$

$\frac{dl}{d\theta} = 0$

$\Rightarrow b\,\text{cosec}\cot\theta = a\sec\theta\tan\theta$

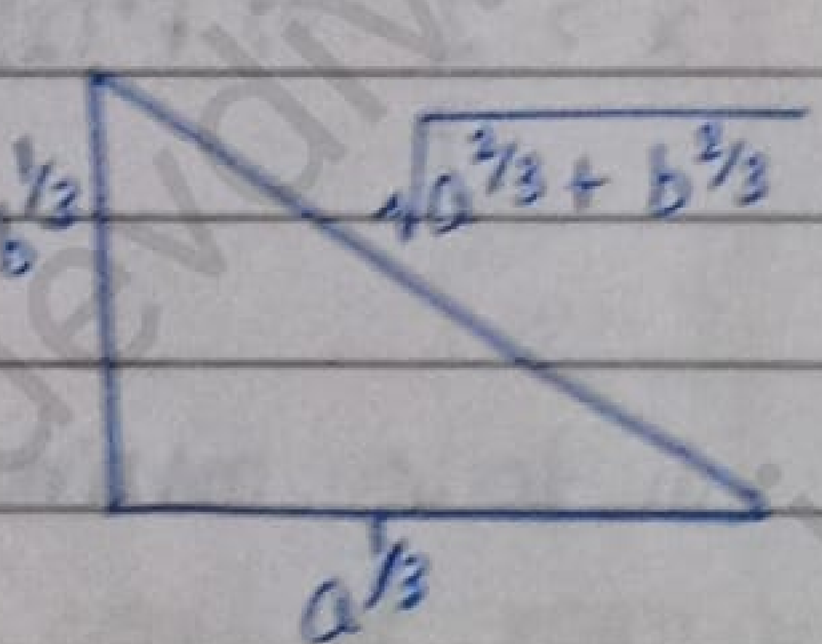
$\Rightarrow \tan^3\theta = \frac{b}{a}$

$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{1/3}$

$\frac{d^2l}{d\theta^2} = -b[-\text{cosec}\,\theta\cot\theta\cot\theta + \text{cosec}\,\theta \times -\text{cosec}^2\theta] + a[\sec\theta\tan\theta\tan\theta + \sec\theta \times \sec^2\theta]$

$= b\,\text{cosec}\,\theta\cot^2\theta + b\,\text{cosec}^3\theta + a\sec\theta\tan^2\theta + a\sec^3\theta$

$\left.\frac{d^2l}{d\theta^2}\right]_{\text{at } \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}} > 0$



$l = b\,\text{cosec}\,\theta + a\sec\theta$

$= \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} + \frac{a \times \sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$

$= \sqrt{a^{2/3}+b^{2/3}}\,(b^{2/3}+a^{2/3})$

$= (a^{2/3}+b^{2/3})^{3/2}$

Hence, proved

Q10.) Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflection

Sol. $f(x) = (x-2)^4 (x+1)^3$

classmate

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$f'(x) = 4(x+1)^3 (x-2)^3 + 3(x-2)^4 (x+1)^2$

$= (x-2)^3 (x+1)^2 [4(x+1) + 3(x-2)]$

$= (x-2)^3 (x+1)^2 [4x+4+3x-6]$

$= (x-2)^3 (x+1)^2 (7x-2)$

$f'(x) = 0$

$x = 2, -1, \frac{2}{7}$

$x < 2 \Rightarrow f'(x) < 0$
$x > 2 \Rightarrow f'(x) > 0$] Negative to positive

$x < -1 \Rightarrow f'(x) > 0$
$x > -1 \Rightarrow f'(x) > 0$] Same

$x < \frac{2}{7} \Rightarrow f'(x) > 0$
$x > \frac{2}{7} \Rightarrow f'(x) < 0$] Positive to negative

(i) Local maxima $\Rightarrow \frac{2}{7}$

(ii) Local minima $\Rightarrow 2$

(iii) Point of inflection $\Rightarrow -1$

Ans

Q11.) Find the absolute maximum and minimum values of the function f given by

$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi]$

Sol. $f'(x) = -2\cos x \sin x + \cos x$

$\cancel{\boxed{-\sin 2x + \cos x}}$

$= \cos x(-2\sin x + 1)$

$f'(x)=0$

$\cos x\ (-2\sin x+1)=0$

$\cos x=0$, $\sin x=\frac{1}{2}$

$x=\frac{\pi}{2}$, $x=\frac{\pi}{6}$, $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$f(0)=\cos^2 0+\sin 0=1$

$f\left(\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}+\sin\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$

$f\left(\frac{\pi}{2}\right)=\cos^2\frac{\pi}{2}+\sin\frac{\pi}{2}=1$

$f\left(\frac{5\pi}{6}\right)=\cos^2\frac{5\pi}{6}+\sin\frac{5\pi}{6}=\left(\frac{-\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$

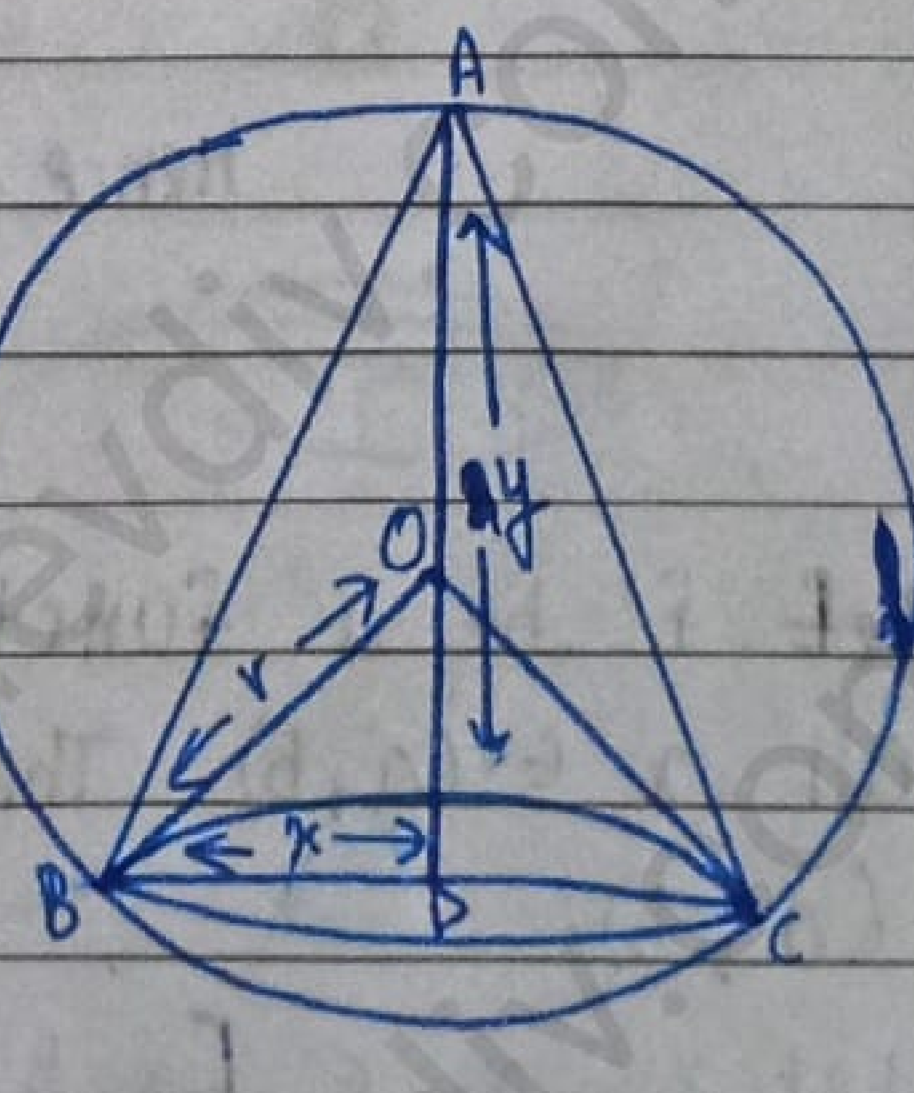
$f(\pi)=\cos^2\pi+\sin\pi=1$

Absolute maximum value $=\frac{5}{4}$

Absolute minimum value $=1$

Ans

Q12) Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.



Sol. $OD=AD-AO=y-r$

In ΔOBD,

$(OD)^2+(BD)^2=OB^2$

$\Rightarrow (y-r)^2+x^2=r^2$

$\Rightarrow y^2+r^2-2ry+x^2=r^2$

$\Rightarrow x^2=2ry-y^2$

Volume of cone $= \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi(2ry - y^2)y$

$\Rightarrow V = \frac{\pi}{3}(2ry^2 - y^3)$

$\frac{dV}{dy} = \frac{\pi}{3}(4ry - 3y^2)$

$\frac{d^2V}{dy^2} = \frac{\pi}{3}(4r - 6y)$

$\frac{dV}{dy} = 0$

$\Rightarrow \frac{\pi}{3}(4ry - 3y^2) = 0$

$\Rightarrow \frac{\pi y}{3}(4r - 3y) = 0$

$\Rightarrow 4r - 3y = 0$

$\Rightarrow y = \frac{4r}{3}$

$\left.\frac{d^2V}{dy^2}\right]_{at\ y = \frac{4r}{3}} = \frac{\pi}{3}\left[4r - \cancel{6}^{2} \times \frac{4r}{\cancel{3}}\right] = \frac{\pi}{3}(-4r) < 0$

$\therefore$ Volume is maximum at $y = \frac{4r}{3}$

Hence, proved

Q13.) Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

Sol. Let $x_1, x_2 \in [a, b]$

such that $x_1 < x_2$

As f is defined everywhere in $[x_1, x_2]$, f is continuous and differentiable in $[x_1, x_2]$

By mean value theorem,
there exists c in (x_1, x_2)
such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

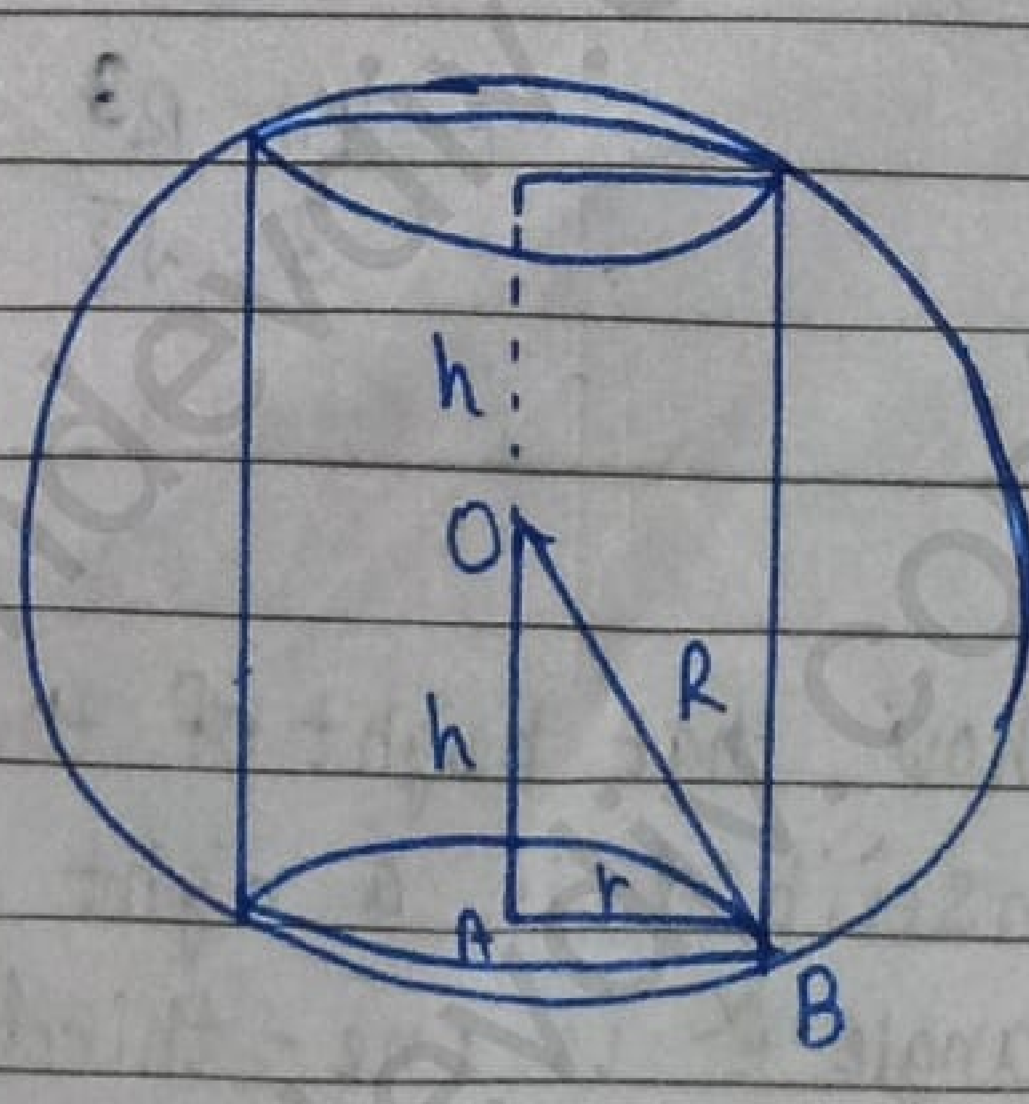
$f'(x) > 0 \quad \forall x \in (a, b)$
$f'(c) > 0 \quad \forall c \in (x_1, x_2)$
$\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$
$f(x_2) - f(x_1) > 0$

For $x_1, x_2 \in [a, b]$
$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

$\therefore$ f is increasing in the interval $[a, b]$
Hence, proved

Q14) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\dfrac{2R}{\sqrt{3}}$. Also find the maximum volume.

Sol. Height of cylinder $= 2h$
Radius $= r$



Radius of sphere $= R$
- $OB^2 = OA^2 + OB^2$

$\Rightarrow R^2 = h^2 + r^2$

$V = \pi r^2 (2h)$

$\Rightarrow V = \pi (R^2 - h^2) 2h$

$\Rightarrow V = 2\pi (R^2 h - h^3)$

$\frac{dV}{dh} = 2\pi (R^2 - 3h^2)$

$\frac{d^2V}{dh^2} = -2\pi 6h < 0$

Volume is maximum

$\frac{dV}{dh} = 0$

$\Rightarrow 2\pi (R^2 - 3h^2) = 0$

$\Rightarrow R^2 = 3h^2$

$\Rightarrow h = \frac{R}{\sqrt{3}}$

$\Rightarrow 2h = \frac{2}{\sqrt{3}} R$ $\boxed{\text{Proved}}$

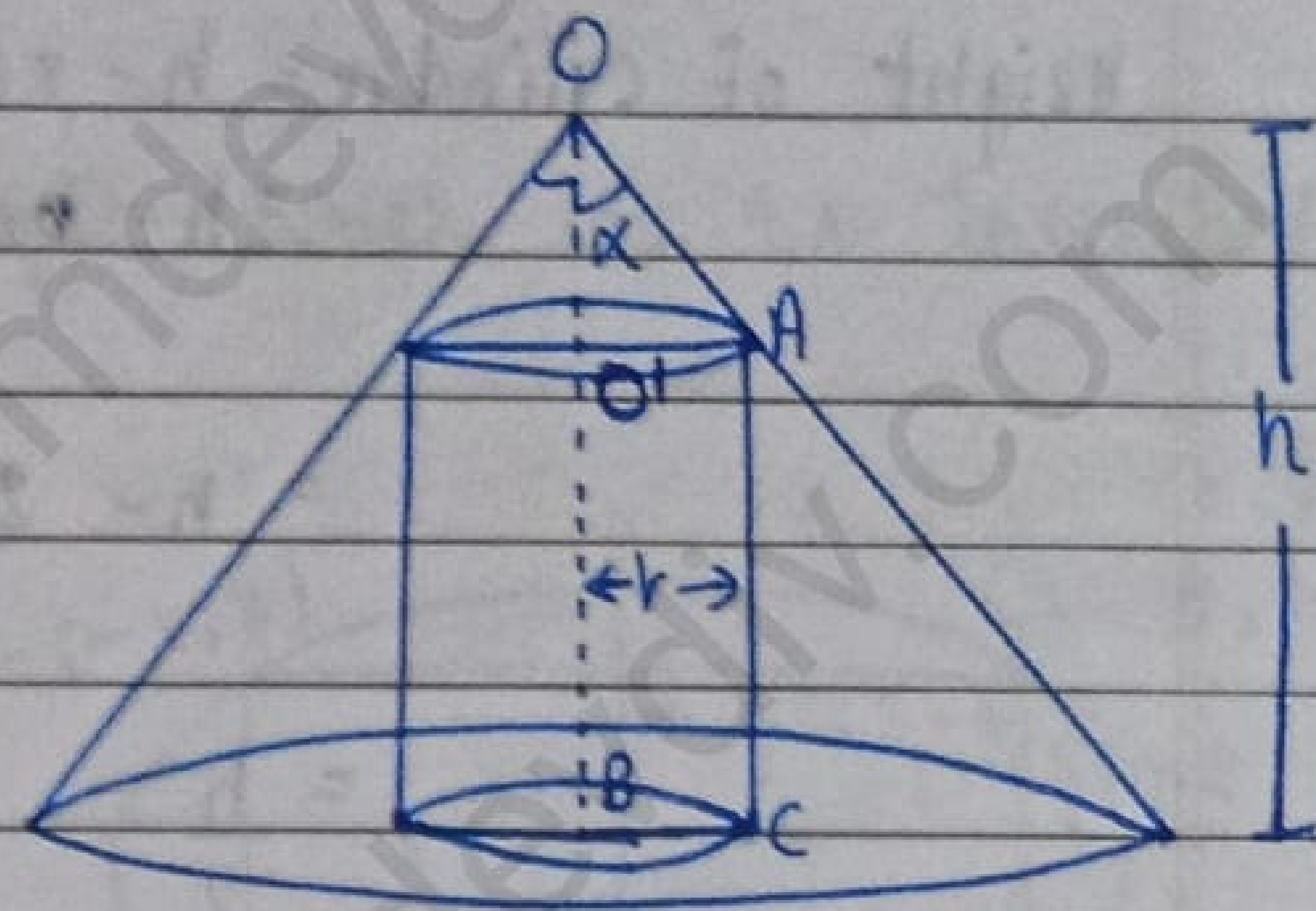
$V = \pi r^2 (2h) = \pi r^2 \times \frac{2}{\sqrt{3}} R$

$= \pi \left[R^2 - \frac{R^2}{3}\right] \frac{2}{\sqrt{3}} R$

$= \frac{\pi 2R^2}{3} \times \frac{2R}{\sqrt{3}}$

$= \frac{4\pi}{3} \frac{R^3}{\sqrt{3}}$ Ans

Q15.) Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.



Sol. $\tan\alpha = \frac{O'A}{OO'}$

$\Rightarrow \tan\alpha = \frac{r}{OO'}$

$\Rightarrow OO' = \frac{r}{\tan\alpha}$

$\Rightarrow OO' = r\cot\alpha$

$O'B = OB - OO' = (h - r\cot\alpha)$

$V = \pi (BC)^2 (O'B)$
$= \pi r^2 (h - r\cot\alpha)$
$= \pi (r^2 h - r^3 \cot\alpha)$

$\frac{dV}{dr} = \pi (2rh - 3r^2 \cot\alpha)$

$\frac{d^2V}{dr^2} = \pi (2h - 6r\cot\alpha)$

$\frac{dV}{dr} = 0$

$\Rightarrow \pi(2rh - 3r^2\cot\alpha) = 0$

$\Rightarrow 2\cancel{r}h = 3r^{\cancel{2}}\cot\alpha$

$\Rightarrow r = \frac{2h}{3\cot\alpha}$

$\left.\frac{d^2V}{dr^2}\right]_{at\ r = \frac{2h}{3\cot\alpha}} = \pi\left[2h - \overset{2}{\cancel{6}} \times \frac{2h}{\cancel{3}\cancel{\cot\alpha}} \times \cancel{\cot\alpha}\right]$

$= -2\pi h < 0$

Volume is maximum

Height of cylinder $= h - r\cot\alpha$

$$= h - \cot\alpha \times \frac{2h}{3\cot\alpha}$$

$$= h - \frac{2h}{3}$$

$$= \frac{h}{3}$$

Height of cylinder is one third of cone $\boxed{\text{Proved}}$

$$V = \pi r^2 (h - r\cot\alpha)$$

$$= \pi\left(\frac{2h}{3\cot\alpha}\right)^2\left(h - \frac{2h}{3}\right)$$

$$= \frac{\pi \times 4h^2}{9\cot^2\alpha} \times \frac{h}{3}$$

$$= \frac{4}{27}\pi h^3 \cdot \tan^2\alpha \quad \boxed{\text{Proved}}$$

Q16.) A cylindrical tank of radius 10m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h

Sol. $V = \pi r^2 h$

$$\frac{dV}{dt} = 314 \ m^3/h$$

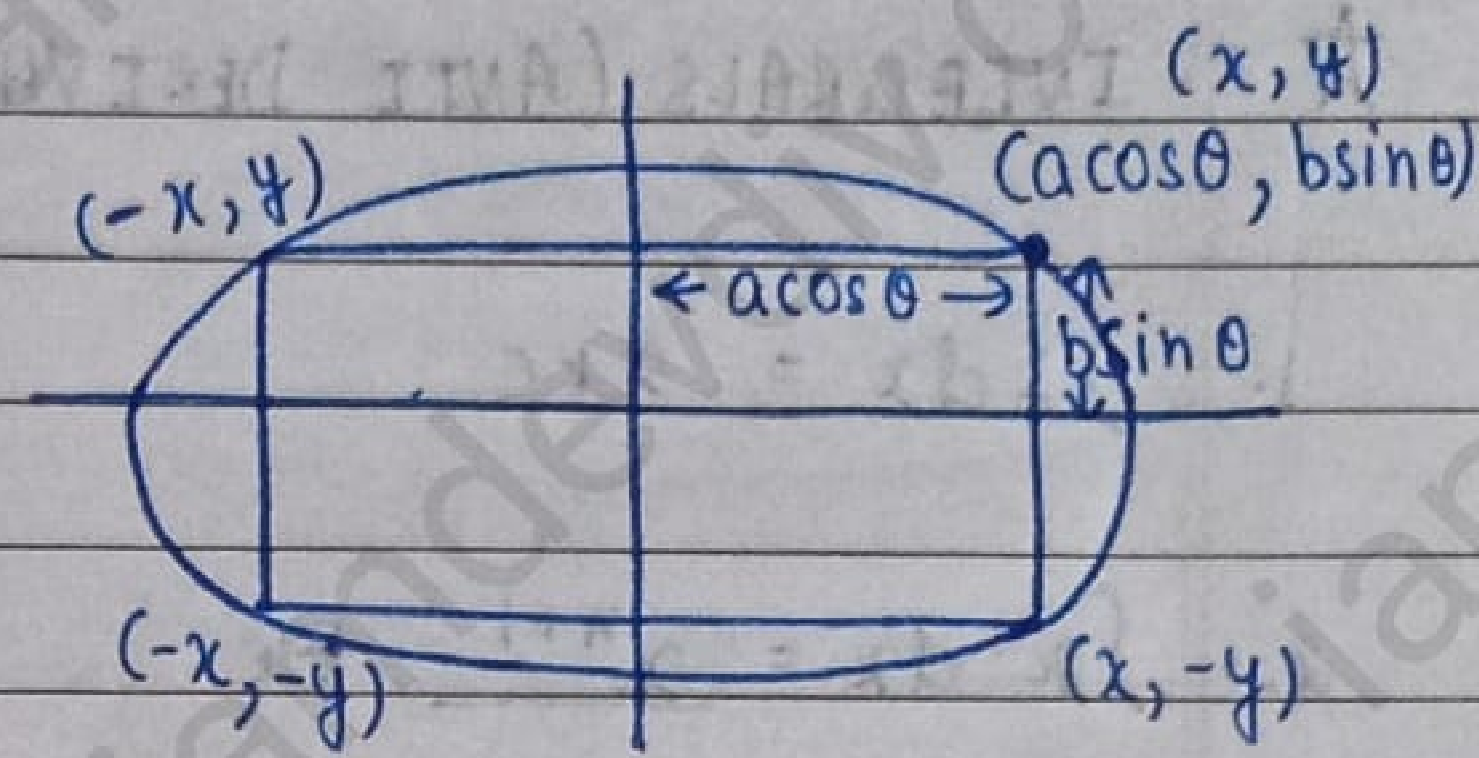
$$\Rightarrow \pi r^2 \frac{dh}{dt} = 314$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100 \times 3.14} = 1 \ m/h$$

$\therefore$ (A) 1 m/h Ans

EXTRA QUESTION

Q Find the area of greatest rectangle which can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Sol. $x = a\cos\theta$

$y = b\sin\theta$

$A = 2a\cos\theta \times 2b\sin\theta$

$\frac{dA}{d\theta} = 2b\sin\theta \times 2a(-\sin\theta) + 2a\cos\theta + 2b\cos\theta$

$= -4ab\sin^2\theta + 4ab\cos^2\theta$

$= 4ab(\cos^2\theta - \sin^2\theta)$

$\frac{d^2A}{d\theta^2} = 4ab(2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta)$

$= 4ab(-2\sin\theta\cos\theta - 2\sin\theta\cos\theta)$

$= -4ab(2\sin 2\theta) < 0$

Area is maximum

$\frac{dA}{d\theta} = 0$

$\Rightarrow 4ab(\cos^2\theta - \sin^2\theta) = 0$

$\Rightarrow \cos^2\theta = \sin^2\theta$

$\Rightarrow \theta = \frac{\pi}{4}$

$A = 2a\cos\theta \times 2b\sin\theta$

$= \frac{2a}{\sqrt{2}} \times \frac{2b}{\sqrt{2}}$

$= 2ab$ Ans